A NOTE ON A CELLULARITY THEOREM BY DOYLE

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A continuum $K$ in $S^n$ is cellular if $S^n\setminus K$ is topologically $E^n$. In [2] Doyle gave a sufficient condition for cellularity of an arc in $S^n$. In this paper we show that this condition is sufficient for cellularity of dendrites and can be used to get a condition for cellularity of unions of dendrites with cellular sets in $S^n$.

A continuum $K$ in $S^n$ is said to be $\wp$-shrinkable if there is a point $q$ of $K$ and for each open set $U$ containing $q$ there is a closed $n$-cell $C \subseteq U$ such that $q$ lies in $\text{Int } C$ and $K$ meets $\text{Bd } C$ in exactly one point. If $y$ is a point of $K$ different from $q$, we will say that $K$ is $\wp$-shrinkable towards $y$. The proofs of the following lemma and theorem are identical to those given by Doyle for Lemma 1 and Theorem 1 in [2] and, therefore, will not be given here.

**Lemma 1.** Let $C$ be a closed $n$-cell and $N$ a continuum which lies in $\text{Int } C$ except for a point $x$ of $N$ which lies on $\text{Bd } C$. Then there is a pseudo-isotopy of $C$ onto $C$ which is fixed on $\text{Bd } C$ and which carries $N$ onto $x$.

**Theorem 1.** Let $K$ be a continuum in $S^n$ which contains a cellular subset $L$. Suppose that each subcontinuum $K'$ of $K$ which contains $L$ as a proper subcontinuum is $\wp$-shrinkable. Then, $K$ is cellular.

**Corollary 1.** If each subcontinuum of a dendrite in $S^n$ is $\wp$-shrinkable, then the dendrite, as well as each of its subcontinua, is cellular.

**Corollary 2.** Suppose $K$ is a dendrite, $y \in K$, and $L$ is a cellular set in $S^n$. Suppose also that $K \cap L = \{y\}$ and that each subcontinuum of $K$ containing $y$ is $\wp$-shrinkable towards $y$. Then, $K \cup L$ is cellular.

It is worthwhile to point out here that it is not enough to require that each arc of a dendrite be $\wp$-shrinkable in order to get cellularity of the dendrite. If we consider a dendrite on the Alexander horned sphere whose end points are the “bad” points of the horned sphere, we get a noncellular dendrite, each of whose arcs is tame and, therefore, $\wp$-shrinkable. If the dendrite has only a countable number of end points, then it would be sufficient to have each arc $\wp$-shrinkable. Using the concept of being $\wp$-shrinkable, we get a local condition which gives cellularity. This condition, however, is not necessary.

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There are cellular arcs in $S^3$ which pierce no disk. Consider any arc of the simple closed curve described in [1]. Each subarc of such an arc would fail to be $\varphi$-shrinkable.

To see that an arc of this simple closed curve is cellular, let \( \{ T_i \}_{i=1}^{\infty} \) be the nested sequence of solid tori which has the simple closed curve as its intersection. From \( T_1 \) remove a thickened disk \( C \) such that \( \text{Cl}(T_1 \setminus C) = C_1 \) is a 3-cell. Let \( C_2 \) be the 3-cell of \( T_2 \cap C_1 \) of largest diameter. In general let \( C_{n+1} \) be the 3-cell of \( T_{n+1} \cap C_n \) of largest diameter. Let \( \alpha = \bigcap_{i=1}^{\infty} C_i \). Since each \( C_i \) is tame, \( \alpha \) is cellular.

References


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