ON MINIMAL SEPARATING COLLECTIONS

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1. **Introduction.** Minimal separating collections were introduced in [2, §8]; their knowledge is useful for the computation of the kernel of a cooperative game. In this note we determine an exact bound on the maximum number of sets which a minimal separating collection can have. The proof makes use of a result on finite graphs, which is proved in the next section.

2. **Minimally ordered graphs.** Throughout this paper we deal only with directed graphs. Our terminology is that of [1]. We recall some definitions that pertain to our work. A *finite graph* G is a pair (X, Γ), where X is a finite set and Γ is a multivalued function mapping X into X, i.e., for each $x \in X$, $\Gamma(x)$ is a subset of X. Let $G = (X, \Gamma)$ be a finite graph. An element x of X is called a *vertex*. The number of vertices of X is denoted by n. An ordered pair of vertices (x, y) with $y \in \Gamma(x)$ is an *arc*. A *path* is a sequence of vertices $\mu = [x_1, \cdots, x_{k+1}]$ such that $x_{i+1} \in \Gamma(x_i)$ for $i = 1, \cdots, k$. Let $\mu = [x_1, \cdots, x_{k+1}]$ be a path. μ is a *circuit* if $x_1 = x_{k+1}$; if, in addition, $x_j \neq x_i$ for $i \neq j$, $1 \leqq i, j \leqq k$, then μ is an *elementary circuit*. G is *strongly connected* if for every ordered pair of distinct vertices x and y there exists a path $\mu = [x, a_1, \cdots, a_{k-1}, y]$. A *partial graph* G' of G is a graph (X, Γ'), where $\Gamma'(x) \subset \Gamma(x)$ for every $x \in X$; G' is a *proper partial graph* of G if there exists a vertex y such that $\Gamma'(y)$ is a proper subset of $\Gamma(y)$. Let A be a subset of X. The *subgraph* of G determined by A is the graph (A, Γ_A) where $\Gamma_A(a) = \Gamma(a) \cap A$, for all $a \in A$. An s-graph is a set Y together with a collection U of ordered pairs of members of Y; U may contain the same ordered pair as many as s times. Clearly a 1-graph is a graph. *The shrinkage* of A is the s-graph obtained from G by deleting the arcs of the subgraph (A, Γ_A), and by identifying all the vertices of A. The number of members of A is denoted by $|A|$. Let x and y be vertices of G. We write $x \leqq y$ if $x = y$ or if there exists a path $\mu = [x, a_1, \cdots, a_{k-1}, y]$. The relation $\geqq$ is *the weak ordering associated with* G. We write $x < y$ if $x \leqq y$ but not $x \geqq y$; we write $x \equiv y$ if $x \leqq y$ and $y \leqq x$. The relation $\equiv$ is the equivalence relation derived from $\leqq$.

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DEFINITION 2.1. A finite graph G is *minimally ordered* if it has no proper partial graph which defines the same weak ordering on the vertices of G as G.

Clearly a minimally ordered strongly connected graph is minimally connected [**1**, p. 123].

LEMMA 2.2. *A subgraph of a minimally ordered graph is minimally ordered.*

The proof, which is straightforward, is omitted.

LEMMA 2.3. *Let G be a minimally ordered graph and let $A \subset X$ determine a strongly connected subgraph; the shrinkage of A leads to a minimally ordered graph.*

The proof, which is similar to the proof of Theorem 1 [**1**, p. 123], is omitted.

LEMMA 2.4. *A minimally connected graph G has at most $2(n-1)$ arcs.*

PROOF. The proof is by induction on n. For $n=1$ the lemma is true. Let G be a minimally connected graph with n vertices, $n \geqq 2$. G contains an elementary circuit μ with l vertices, $l \geqq 2$. By Theorem 1 [**1**, p. 123], the shrinkage of μ leads to a minimally connected graph G'. G' has $n-l+1$ vertices. Since $n-l+1<n$, G' has, by assumption, no more than $2(n-l)$ arcs. By Theorem 2 [**1**, p. 124] G has at most $2(n-l)+l=2n-l \leqq 2n-2$ arcs.

EXAMPLE 2.5. Let X be a set with n members, $n \geqq 1$, and let $a \in X$. Define $\Gamma(a)=X-\{a\}$ and $\Gamma(x)=\{a\}$, for $x \in X-\{a\}$. (X, Γ) is a minimally connected graph with $2(n-1)$ arcs.

LEMMA 2.6. *A minimally ordered graph without circuits has at most $f(n)$ arcs, where $f(n)=\frac{1}{4}n^2$ if n is even, and $f(n)=\frac{1}{4}(n^2-1)$ if n is odd.*

PROOF. The proof is by induction on n. For $n=1$ the lemma is true. Let $G=(X, \Gamma)$ be a minimally ordered graph without circuits with n vertices. We shall assume that n is even. The proof for odd n is similar. Let a be a minimal vertex of G, i.e., there exists no $x \in X$ such that $a>x$. Let $T=\{b \mid b>a$ and there exists no c such that $b>c>a\}$. We remark that $x \in \Gamma(a)$ if and only if $x \in T$. If $|T| \leqq \frac{1}{2}n$ let $X_1 = X-\{a\}$. If $|T|>\frac{1}{2}n$ let $b \in T$ and $X_1=X-\{b\}$. We remark that the number of arcs incident into or out from b is less than $\frac{1}{2}n$, since if $b_1 \in T$ then neither $b_1>b$ nor $b>b_1$. By Lemma 2.2 the subgraph determined by X_1 is minimally ordered; since it has no circuits it has, by assumption, at most $f(n-1)$ arcs. Hence G has no more than $f(n-1)+\frac{1}{2}n=f(n)$ arcs.

EXAMPLE 2.7. Let X be a set with n members, $n \geqq 1$. Let $A \subset X$ have $\frac{1}{2}n$ members if n is even, and $\frac{1}{2}(n+1)$ members if n is odd. Let $\Gamma(a) = X - A$, for $a \in A$. (X, Γ) is a minimally ordered graph without circuits which has $f(n)$ arcs.

LEMMA 2.8. *A minimally ordered graph $G = (X, \Gamma)$ has at most $g(n)$ arcs, where $g(n) = \max(2(n-1), f(n))$.*

PROOF. Let $X_1, \cdots, X_k$ be the equivalence classes determined by the equivalence relation derived from the weak ordering associated with G. By Lemma 2.2, for $i = 1, \cdots, k$, (X_i, Γ_{X_i}) is a minimally connected subgraph of G. If $k = n$ then G has no circuits. Thus if $k = 1$ or $k = n$ the lemma is true by Lemmas 2.4 and 2.6. If $2 \leqq k \leqq n-1$ then, by Lemma 2.3, the shrinkage of $X_1, \cdots, X_k$ leads to a minimally ordered graph G^*. Clearly G^* has no circuits; hence it has no more than $\frac{1}{4}k^2$ arcs. By Lemma 2.4 each of the subgraphs (X_i, Γ_{X_i}), $i = 1, \cdots, k$, has at most $2(|X_i| - 1)$ arcs. Thus G has no more than

$$\tfrac{1}{4}k^2 + 2\sum_{i=1}^{k}(|X_i| - 1) = \tfrac{1}{4}k^2 + 2(n-k) \leqq g(n)$$

arcs.

COROLLARY 2.9. *Every finite graph G has a partial graph with no more than $g(n)$ arcs which defines the same weak ordering on the vertices of G as G.*

3. **Minimal separating collections.** We recall some of the definitions of [2]. Let N be a finite nonempty set. Let $\mathfrak{D}$ be a collection of subsets of N and let $i, j \in N$, $i \neq j$. i is separated by $\mathfrak{D}$ from j if there exists a set $S \in \mathfrak{D}$ such that $i \in S$ and $j \notin S$. $\mathfrak{D}$ is *separating* if for every pair $i, j \in N$, i is separated from j by $\mathfrak{D}$ if and only if j is separated from i by $\mathfrak{D}$. A separating collection is *minimal separating* if it does not contain a proper subcollection which is separating. A collection $\mathfrak{D}$ is *completely separating* if for all pairs $i, j \in N$, $i \neq j$, i is separated from j and j is separated from i by $\mathfrak{D}$. A completely separating collection is *minimal completely separating* if it does not contain a proper subcollection which is completely separating. We remark that a completely separating collection which is minimal separating is a minimal completely separating collection.

EXAMPLE 3.1. Let N be a finite nonempty set. The collections $\mathfrak{D}_1(N) = \{\{i\} : i \in N\}$ and $\mathfrak{D}_2(N) = \{N - \{i\} \mid i \in N\}$ are minimal separating and completely separating.

EXAMPLE 3.2. Let $N = \{1, \cdots, n\}$ be the set of the first n natural numbers. Let $S_i = \{j \mid j \leqq i\}$, $i = 1, \cdots, n-1$. The collection

$\{S_1, \cdots, S_{n-1}, N-S_1, \cdots, N-S_{n-1}\}$ is minimal completely separating and has $2(n-1)$ sets.

EXAMPLE 3.3. Let N be a set with n members, $n \geqq 5$. Let $A \subset N$ have $\frac{1}{2}n$ members if n is even, and $\frac{1}{2}(n-1)$ members if n is odd. The collection $\{S \cup T \mid S \in \mathfrak{D}_1(A),\ T \in \mathfrak{D}_2(N-A)\}$ is minimal separating and completely separating and has $f(n)$ sets (see Example 3.1, where $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are defined, and Lemma 2.6 where $f(n)$ is defined).

DEFINITION 3.4. Let $\mathfrak{D}$ be a collection of subsets of a finite nonempty set N. An ordered pair (i, j) of members of N is a *distinguished pair* (with respect to $\mathfrak{D}$) if there is *exactly one* set $S \in \mathfrak{D}$ such that $i \in S$ and $j \notin S$.

LEMMA 3.5. *If $\mathfrak{D}$ is a minimal completely separating collection of subsets of a finite nonempty set N, then for each $S \in \mathfrak{D}$ there exists a distinguished pair (i, j) such that $i \in S$ and $j \notin S$.*

The proof, which is straightforward, is omitted.

THEOREM 3.6. *Let N be a finite nonempty set with n members. The maximum number of sets in a minimal completely separating collection of subsets of N is $g(n)$ (see Lemma 2.8 where $g(n)$ is defined).*

PROOF. Let $\mathfrak{D}$ be a minimal completely separating collection of subsets of N. Define a graph $G=(N, \Gamma)$, where Γ is defined by: $j \in \Gamma(i)$ if and only if (i, j) is a distinguished pair (with respect to $\mathfrak{D}$). By Corollary 2.9, G has a partial graph $G'=(N, \Gamma')$ which defines the same weak ordering as G and has no more than $g(n)$ arcs. Let $\mathfrak{D}' \subset \mathfrak{D}$ be defined by: $\mathfrak{D}' = \{S \mid S \in \mathfrak{D}$ and there exists a pair (i, j) such that $j \in \Gamma'(i),\ i \in S$ and $j \notin S\}$. $\mathfrak{D}'$ has at most $g(n)$ sets. We shall show that $\mathfrak{D}' = \mathfrak{D}$. Suppose, per absurdum, that there exists $S \in \mathfrak{D} - \mathfrak{D}'$. By Lemma 3.5 there exists a distinguished pair (i, j) such that $i \in S$ and $j \notin S$; $j \geqq i$ according to the ordering of G. Since G' defines the same ordering, there exists a path $\mu = [i, a_1, \cdots, a_{l-1}, j]$ in G'. Let $S_1, \cdots, S_l$ be sets in $\mathfrak{D}'$ such that $i \in S_1$, $a_1 \notin S_1$, $a_1 \in S_2$, $a_2 \notin S_2, \cdots,$ $a_{l-1} \in S_l$, $j \notin S_l$. Since $i \in S$, $S \neq S_1$ and (i, a_1) is a distinguished pair, $a_1 \in S$. Using induction we can show that all the vertices of μ are in S. Since $j \notin S$ we have a contradiction which shows that $\mathfrak{D}' = \mathfrak{D}$. Hence the number of sets in $\mathfrak{D}$ is not greater than $g(n)$. Examples 3.2 and 3.3 show that the bound $g(n)$ is attained.

COROLLARY 3.7. *Let N be a finite nonempty set with n members. The maximum number of sets in a minimal separating completely separating collection of subsets of N is $f(n)$ for $n \geqq 7$, and is not greater than $2(n-1)$ for $n < 7$.*

PROOF. Theorem 3.6 and Example 3.3.

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IMBEDDING CLOSED RIEMANN SURFACES IN C^n

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I. **Introduction.** Let R be a closed Riemann surface of genus g, G a nonempty, open subset of R, and A the set of all complex valued functions that are continuous on R and holomorphic on G. With the usual pointwise operations A is an algebra over the complex field. We consider the problem: how many functions in A suffice to separate points of R?

Let f be a nonconstant member of A. If the genus $g=0$, Wermer [**4**] showed that there exist f_1 and f_2 in A which, together with f separate points of R; if $g=1$, Arens [**2**] established the existence of f_1, f_2 and f_3 in A which, together with f separate points of R. In this note we shall present a modification of the Wermer-Arens argument to prove the following

THEOREM. *Let the genus g be arbitrary. If A contains nonconstant functions, then there exist four functions in A which separate points of R and which have no common branch points in G.*

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II. **Two lemmas.** Let ϕ be a nonconstant member of order n in the field K of meromorphic functions on R. Let w be a point of the extended plane which has n distinct inverse images under ϕ. Denote by $E(\phi, w)$ the finite set which is the union of $\phi^{-1}(w)$ and $\phi^{-1}(\phi(b))$ as b ranges over all the branch points of ϕ. For (fixed) ϕ and ψ in K, let S

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