THE TOTAL GROUP OF A GRAPH[1]

MEHDI BEHZAD AND HEYDAR RADJAVI

We consider ordinary graphs (finite, undirected, with no loops or multiple lines). A well-known concept in the theory of graphs is that of the (point) group $\Gamma(G)$ of a graph G, which is the group of all adjacency-preserving permutations of points of G. The elements of $\Gamma(G)$ are called (point-) automorphisms of G. In contrast with the notion of $\Gamma(G)$ is that of the *line group* $\Gamma'(G)$ of G consisting of all adjacency-preserving permutations of lines of G. This group has been considered in [**6**]. The purpose of this paper is to treat another natural concept, called *total group*, and to prove that for any graph G having more than one point, the total group of G is isomorphic to the group of G if and only if no component of G is either a cycle or a complete graph.

1. **Preliminaries.** Denote the point set of G by $V(G)$ and its line set by $X(G)$. Each member of $V(G)\cup X(G)$ will be called an *element* of G. We say two elements of G are *associated* if they are either adjacent or incident. The group of all permutations of elements of G which preserve association will be called the *total group* of G and denoted by $\Gamma''(G)$.

The *line graph* [**6**] of a graph G, denoted by $L(G)$, is that graph whose point set is $X(G)$, and in which two points are adjacent if and only if they are adjacent in G. It is worth observing that $\Gamma'(G) \simeq \Gamma(L(G))$. The notion of *total graphs* introduced by one of the authors [**1**] is a convenient tool for our purposes. The total graph $T(G)$ of G is that graph whose point set is $V(G)\cup X(G)$, and in which two points are adjacent if and only if they are associated in G. We should note that $\Gamma''(G)$ is isomorphic to $\Gamma(T(G))$. The graphs G and $L(G)$ are disjoint subgraphs of $T(G)$. For illustration two graphs G and H are given in Figure 1 together with their line and total graphs. We observe that $\Gamma(G)\simeq\Gamma(T(G))\simeq S_2$ (the symmetric group on two letters) while $\Gamma(H)\simeq S_2$ and $\Gamma(T(H))\simeq S_3$.

The number of lines incident with a point v of a graph G is denoted by $\deg_G(v)$. For a point v of $T(G)$ belonging to $V(G)$ we have $\deg_{T(G)}(v) = 2\deg_G(v)$, and for a point u of $T(G)$ belonging to $X(G)=V(L(G))$ we have $\deg_{T(G)}(u)=\deg_G(a)+\deg_G(b)$, where a and b are the points of G incident with u. Thus if $\deg_{T(G)}(v)$ is maximal, then it is an even number.

Received by the editors November 18, 1966.

[1] Definitions not given here may be found in [**3**] or [**4**].

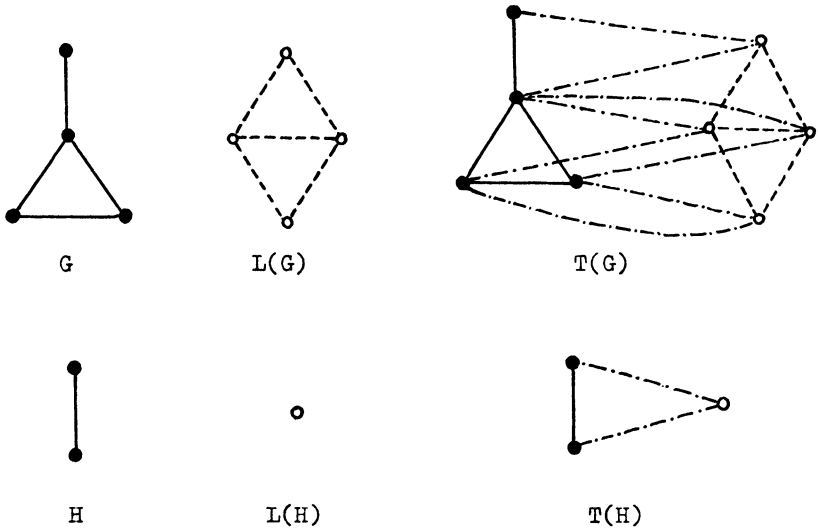

FIGURE 1

If a is a point of a graph G, then the subset of $V(G)$ consisting of the points adjacent with a is called the *neighborhood* of a in G and is denoted by $N_G(a)$. The subgraph of G generated by the points a_1, $a_2, \cdots, a_n$ of G will be denoted by $\langle a_1, a_2, \cdots, a_n\rangle_G$. Finally, we denote by K_p the complete graph of order p.

2. **Results.** We begin this section with

LEMMA 1. *Assume G is any connected graph which is not a path (arc), a cycle, or a complete graph. Let a_0 be a point of $H=T(G)$ which has maximal degree $2d$. Then $a_0 \in V(G)$ if and only if the graph generated by $N_H(a_0)$ has exactly one K_d as a subgraph.*

PROOF. The hypotheses imply that $d \geqq 3$.

(i) Assume $a_0 \in V(G)$. The set $N_H(a_0)$ consists of d points a_i of G and d points b_i of $L(G)$, $i=1, 2, \cdots, d$. Since the b_i are exactly those points of $L(G)$ which correspond to the lines a_0a_i of G, which are all mutually adjacent in G, it follows that the b_i are all mutually adjacent in H and generate a K_d in H. Now each b_i is, by definition of total graphs, adjacent with exactly one of the a_i, $1 \leqq i \leqq d$, and therefore no subset of $N_H(a_0)$ which contains some a_i together with some b_i can generate a K_d. Nor can the a_i themselves generate a K_d. For, otherwise $\langle a_0, a_1, \cdots, a_d\rangle_H = K_{d+1}$, and since G is connected but not complete, there should exist at least one more point a_{d+1} in G adjacent

with some a_j—contradicting the maximality of d. Thus the necessity of the condition is established.

(ii) Assume $a_0 \in V(L(G))$. We observe that $N_H(a_0)$ consists of two points a_1 and a_2 of G and $2(d-1)$ points b_i of $L(G)$. The latter points are those in $L(G)$ which correspond to the lines of G incident with a_1 or with a_2; they are, by the maximality of d, distributed equally between $N_H(a_1)$ and $N_H(a_2)$. Those of the b_i which are in $N_H(a_1)$ are all mutually adjacent and generate a K_d together with a_1. Similarly a_2 and the remaining b_i generate a K_d. This completes the proof of the lemma.

THEOREM 1. *Let G be a connected graph which is neither a cycle nor a complete graph. If $H=T(G)$, then G is the only subgraph of H whose total graph is H.*

PROOF. Let G' be any subgraph of H such that $H=T(G')$. We shall show that $G'=G$.

(i) Assume G is not a path. Take a point a_0 of G such that $\deg_G(a_0)=d$ is maximal in G. By Lemma 1 the point a_0 also belongs to G'. We assert that $N_G(a_0)=N_{G'}(a_0)$. This follows from the fact that the d points $b_1, b_2, \cdots, b_d$ of $N_H(a_0)-N_G(a_0)$ form the only K_d in $N_H(a_0)$. By the proof of Lemma 1, the b_i which belong to $L(G)$ should belong to $L(G')$ as well. Let $a_1, a_2, \cdots, a_d$ be the points of $N_G(a_0)$. By the definition of total graphs two points of G can be adjacent in $T(G)$ if and only if they are adjacent in G. Thus $A_1=\langle a_0, a_1, \cdots, a_d\rangle_H$ is contained in both G and G'.

Now we proceed by induction. Assume that for $k \geqq 1$ the points $a_0, a_1, \cdots, a_{d+k-1}$ of H have been determined such that $A_k=\langle a_0, a_1, \cdots, a_{d+k-1}\rangle_H$ is contained in both G and G'. Also assume that the subgraph $L(A_k)$ of H is contained in $L(G')$ as well as in $L(G)$. If $G \neq A_k$, then there exists at least one point a_{d+k} of G which is adjacent in G with some point a_j of A_k. We show that a_{d+k} is also a point of G'. If not, a_{d+k} belongs to $L(G')$ and corresponds to a line of G' incident with a_j, say a_jc. By the induction hypotheses c is not a point of A_k. There is at least one line in A_k incident with a_j, say a_ja; we denote the point of $L(A_k)$ corresponding to this line by b. The points b and a_{d+k} have to be adjacent in $T(G')$; but this implies the contradiction that in $T(G)$ the point b corresponds to the two lines a_ja and a_ja_{d+k}. Thus $A_{k+1}=\langle a_0, a_1, \cdots, a_{d+k}\rangle_H$ is a subgraph of both G and G'.

Let a_i be any point of A_k adjacent with a_{d+k}. Denote by b' the point of $L(G')$ corresponding to $a_{d+k}a_i$. If b' is not in $L(G)$, then it is a point of G. But then the proof, given above, for the assertion that A_{d+k} belongs to G' is applicable to b' and leads to a contradiction. Thus b'

is the point of $L(G)$ corresponding to the line $a_{d+k}a_i$. Hence $L(A_{k+1})$ is a subgraph of both $L(G)$ and $L(G')$. This completes the induction.

If p is the order of G, then $A_{p-d}=G$ implying that G is a subgraph of G' and that $L(G)$ is a subgraph of $L(G')$. This proves at once that $G=G'$.

(ii) Let G be a path of order p. Since G is not complete we have $p\geqq 3$. The graph H has exactly two points a_1 and a_p of degree 2 which are the end points of the path G. Since the degree of no point of H exceeds 4, the degree of no point of G' can exceed 2. Thus G' is a path with end points a_1 and a_p. Observing that the order of G' should also be p, and that G is the only path of order p in H with end points a_1 and a_p, it follows that $G'=G$. This completes the proof of Theorem 1.

Considering disconnected graphs, we note that if G has n components G_i, then $T(G)$ has n components $T(G_i)$, $i=1, 2, \cdots, n$, and vice versa. Thus we have

COROLLARY 1. *Let G be a graph none of whose components is a cycle or a complete graph. If $T(G)=T(G')$ for any subgraph G' of $T(G)$, then $G'=G$.*

The following theorem is of interest in itself; it is the analog of the theorem of Whitney [6] for line-graphs which states that given any two connected graphs G_1 and G_2 other than K_3 and $K_{1,3}$ we have $L(G_1)\simeq L(G_2)$ if and only if $G_1\simeq G_2$. (The symbol $\simeq$ indicates isomorphism.)

THEOREM 2. *Let G_1 and G_2 be two graphs. Then $T(G_1)\simeq T(G_2)$ if and only if $G_1\simeq G_2$.*

PROOF. It suffices to prove the result for connected graphs. The isomorphism of $T(G_1)$ and $T(G_2)$ follows from that of G_1 and G_2 by definition. Thus we show the converse. In view of Theorem 1 we can confine ourselves to the case in which G_1 is either a cycle or a complete graph. Since a graph is regular of degree d and order p if and only if its total graph is regular of degree $2d$ and order $p(1+d/2)$, it follows that in both cases G_2 is regular and has the same degree and order as G_1. This implies, in both cases, that $G_1\simeq G_2$.

Next we state our main result.

THEOREM 3. *For any graph G other than a single point the two groups $\Gamma''(G)$ and $\Gamma(G)$ are isomorphic if and only if no component of G is either a complete graph or a cycle.*

PROOF. Each automorphism f of G can be extended to an automorphism f' of $T(G)$ as follows: if a is a point of $V(T(G))-V(G)$, then

it corresponds to some line bc of G, and we let $f'(a)=a'$, where a' is the point of $T(G)$ corresponding to $f(b)f(c)$. The function f' is indeed an automorphism of $T(G)$, because (1) if $a\in V(T(G))-V(G)$, then it is adjacent with exactly two points a_1 and a_2 of G; by definition of f' the point $f'(a)$ is adjacent to the two points $f'(a_1)$ and $f'(a_2)$, and (2) if a_1 and a_2 are two adjacent points of $V(T(G))-V(G)$, then the adjacency of $f'(a_1)$ and $f'(a_2)$ follows from the fact that the restriction of f' to $L(G)$ is the automorphism induced [5] by f.

If f_1 and f_2 are any two automorphisms of G, we observe that $(f_1f_2)'=f_1'f_2'$. Furthermore, if f_1 and f_2 are distinct, then so are f_1' and f_2', so that $\Gamma(T(G))$ (which is the same as $\Gamma''(G)$) contains an isomorphic image of $\Gamma(G)$.

Let $g\in\Gamma(T(G))$. If we denote by G' the image under g of the subgraph G of $T(G)$, then the image of the subgraph $L(G)$ is a subgraph which is necessarily $L(G')$. Also if v is a point of $L(G)$ adjacent with a point u of G, then $g(v)$ is a point of $L(G')$ adjacent with the point $g(u)$ of G'. Hence $T(G)=T(G')$.

If G has no complete graph or cycle as a component, then $G=G'$ by Corollary 1 of Theorem 1. Thus every automorphism of $T(G)$ is an extension of some automorphism of G, implying $\Gamma''(G)=\Gamma(G)$.

If the above subgraphs G and G' are distinct, then for no automorphism f of G we have $g=f'$. So $\Gamma''(G)$ is not isomorphic with $\Gamma(G)$. To complete the proof of the theorem, it suffices to show that if G is either a cycle or a complete graph other than K_1, then $T(G)$ contains a subgraph G', $G'\neq G$, with $T(G)=T(G')$.

(i) If G is a cycle, then the subgraph $L(G)$ of $T(G)$ is isomorphic to G and we have $T(L(G))=T(G)$.

(ii) Now let $G=K_p$, $p>1$, $H=T(G)$, and let a_0 be a point of the subgraph G of H. Denote the points of $N_G(a_0)$ by $a_1, a_2, \cdots, a_{p-1}$. For each i, $1\leqq i\leqq p-1$, let b_i denote the point of the subgraph $L(G)$ of H which corresponds to the line a_0a_i. We shall prove that $T(G')=H$, where $G'=\langle a_0, b_1, \cdots, b_{p-1}\rangle_H$. Let f be the permutation of $V(H)$ defined by: $f(a_i)=b_i$, $f(b_i)=a_i$ for $1\leqq i\leqq p-1$, and $f(v)=v$ otherwise. The function f is an automorphism of H. In fact, for $1\leqq i\leqq p-1$, the a_i are mutually adjacent and so are the b_i; hence it would suffice to show that if v, $v\neq a_0$, is a point of H different from the a_i and b_i, $1\leqq i\leqq p-1$, which is adjacent with some a_k, then v is adjacent with b_k and conversely. Since v and b_k are both points of $L(G)$, they are adjacent if and only if they correspond to adjacent lines of G. This occurs if and only if a_0a_k and the line of G corresponding to v, say a_ra_s, $1\leqq r<s\leqq p-1$, are adjacent, which is true if and only if either

$r=k$ or $s=k$. Since the only a_i adjacent with v are a_r and a_s, our assertion follows.

REMARK. Let G be either a cycle or a complete graph. The subgraphs of $T(G)$ whose total graphs coincide with $T(G)$ have been enumerated elsewhere [2]. Here we treat these cases briefly. We first observe that if G and G' are any two subgraphs of H such that $H=T(G)=T(G')$, then every isomorphism of G onto G' gives rise to an automorphism of H.

(i) For $G=K_3$, $T(G)$ is the total graph of each of its eight triangles; $\Gamma''(G)$ has order 48.

(ii) For a cycle G of order p, $p>3$, the only other subgraph of $T(G)$ whose total graph is $T(G)$ is $L(G)$; $\Gamma''(G)$ has order $4p$.

(iii) For a complete graph G of order p, $p>3$, there are exactly p other subgraphs G_i of $T(G)$ with $T(G_i)=T(G)$, $i=1, 2, \cdots, p$. Thus the order of $\Gamma''(G)$ is $(p+1)!$.

We conclude the paper with the observation that for a connected graph G, $\Gamma(G)\simeq\Gamma'(G)\simeq\Gamma''(G)$ if and only if G is not (1) a complete graph, (2) a cycle, or (3) the graph K_4 minus one or two lines. This follows from our results together with [6].

BIBLIOGRAPHY

1. Mehdi Behzad and G. Chartrand, *Total graphs and traversability*, Proc. Edinburgh Math. Soc. (2) **15** (1966), 117–120.

2. Mehdi Behzad and Heydar Radjavi, *Structure of regular total graphs* (to appear).

3. F. Harary, *A seminar on graph theory*, Holt, Rinehart, and Winston, New York 1967.

4. O. Ore, *Theory of graphs*, Amer. Math. Soc. Colloq. Publ., Vol. 38, Amer. Math. Soc., Providence R. I., 1962.

5. G. Sabidussi, *Loewy groupoids related to linear graphs*, Amer. J. Math. **76** (1954), 477–487.

6. H. Whitney, *Congruent graphs and the connectivity of graphs*, Amer. J. Math. **54** (1932), 150–168.

PAHLAVI UNIVERSITY, SHIRAZ, IRAN