A NOTE ON THE KOSZUL COMPLEX

J. A. EAGON AND M. M. FRASER

It is well known that the Koszul complex $R_{x:1,\cdots,s}$ of an ideal $I=(x_1, \cdots, x_s)$ over a commutative ring R possesses the structure of an exterior algebra (see [**3**, p. 189]). Use of this structure is avoided in [**1**] and [**5**], but one can give easier proofs of some of the results by using it, e.g. the fact that the ideal I annihilates the homology of the complex (see [**3**]). This note gives an easy proof of a more general result than in paragraph 2 of [**2**] or of a remark in [**5**, p. IV-12]. The extra generality is necessary in the theory of multiplicities and Grothendieck groups developed in [**4**].

Our notation is that of [**1**] and [**2**]. We remark that the complex $E_{x:1,\cdots,s}=R_{x:1,\cdots,s}\otimes_R E$ has the structure of a graded left module over the exterior algebra $R_{x:1,\cdots,s}$. The exterior product will be denoted $\wedge$.

PROPOSITION. *Let R be a commutative Noetherian ring with unit. Let $I=(x_1, \cdots, x_s)$. Let E be a finitely generated module over R. Let C be the complex $E_{x:1,\cdots,s}$. Let $C^{(k)}$ be the subcomplex of C given by*

$$0 \to I^kC_s \to I^{k+1}C_{s-1} \to \cdots \to I^{k+s}C_0 \to 0.$$

Then there exists an integer n such that $C^{(k)}$ is acyclic for all $k>n$. Note that no assumptions concerning finite length are made.

PROOF. Let Z_i be the cycles in C_i. Then $Z_i\cap I^{k+s-i}C_i$ are the cycles in $I^{k+s-i}C_i$. By the Artin-Rees lemma we may choose n such that for k larger than n $Z_i\cap I^{k+s-i}C_i=I(Z_i\cap I^{k+s-i-1}C_i)$, and we may certainly do this uniformly for all i. Let z be a cycle in $I^{k+s-i}C_i$. By the above, $z=\sum_{i=1}^s x_ia_i$ where $a_i\in Z_i\cap I^{k+s-i-1}C_i$.

Let u_i be the element of $(R_{x:1,\cdots,s})_1$ such that $d(u_i)=x_i$ when x_i is considered as an element of $(R_{x:1,\cdots,s})_0$. Let $w \sum_{i=1}^s u_i\wedge a_i$. w is in $I^{k+s-i-1}C_{i+1}$.

$$d(w) = \sum_{i=1}^{s} (d(u_i) \wedge a_i - u_i \wedge d(a_i)) = \sum_{i=1}^{s} (x_ia_i - 0) = z. \qquad \text{Q.E.D.}$$

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UNIVERSITY OF ILLINOIS

THE DISTANCE FROM $U(z)\cdot H^p$ TO 1

D. J. NEWMAN

If $U(z)$ is an inner function, then the set $U(z)\cdot H^p$ of all H^p multiples of $U(z)$ forms a closed subspace of H^p. In this note we compute the H^p distance between the constant function 1 and this closed subspace. It is of course well known that this distance is 0 if and only if $U(z)$ is a constant, i.e. if and only if $|U(0)|=1$. We will prove

THEOREM. dist $(1,\ U(z)\cdot H^p)=(1-|U(0)|^2)^{1/p}, \qquad p\geqq 1.$

PROOF. With

$$f_p(z) = \frac{1-(1-U(z)\overline{U}(0))^{2/p}}{U(z)}$$

we have

$$\|1-U(z)f_p(z)\| = \left(\frac{1}{2\pi}\int_0^{2\pi} |1-U(z)\overline{U}(0)|^2 d\theta\right)^{1/p}$$

$$= \left(\frac{1}{2\pi}\int_0^{2\pi} |U(z)-U(0)|^2 d\theta\right)^{1/p} = (1-|U(0)|^2)^{1/p},$$

so that this distance is surely $\leqq (1-|U(0)|^2)^{1/p}$ and we need only show that $f_p(z)$ is the *closest* function to 1, i.e. that, for all $f(z)\in H^p$,

$$\|1-U(z)f(z)\|_p \geqq (1-|U(0)|^2)^{1/p}. \tag{1}$$

Consider

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