ON A SEMIPRIMARY RING
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Let \( R \) be a ring with 1 having radical (Jacobson) \( N \). \( R \) is called semiprimary [2, p. 56] if and only if \( R/N \) satisfies the minimum condition for right ideals. If \( M \) is a right \( R \)-module, a submodule \( A \) of \( M \) is called small [5] if \( A + B = M \) for any submodule \( B \) of \( M \) implies \( B = M \). A submodule \( A \) of \( M \) is called large [3] if \( A \cap B = 0 \) for any submodule \( B \) of \( M \) implies \( B = 0 \). A right ideal in \( R \) is called small or large if it is small or large as a submodule of the right regular \( R \)-module \( R_R \). A projective cover [1] of \( M \) is an epimorphism of a projective module onto \( M \) such that its kernel is small. The main results of this paper are the following theorems:

**Theorem 1.** Every irreducible (right) \( R \)-module has a projective cover if and only if \( R \) is semiprimary and for any nonzero idempotent \( x+N \) in \( R/N \) there is a nonzero idempotent \( e \) in \( R \) such that \( ex - e \in N \).

Theorem 1 is related to Theorem 2.1 of [1].

**Theorem 2.** If \( R \) is commutative then every irreducible \( R \)-module has a projective cover if and only if \( R \) is semiprimary and for any nonzero idempotent \( x+N \) in \( R/N \) there is an idempotent \( e \in R \) such that \( x - e \in N \).

**Lemma 1.** If \( I \) is a maximal right ideal of \( R \) then the right \( R \)-module \( R/I \) has a projective cover if and only if there is a nonzero idempotent \( e \in R \) such that \( eI \) is small.

**Proof.** Let \( f \) be an epimorphism from a projective module \( P \) onto \( R/I \) such that the kernel of \( f \) is small in \( P \). Since \( R \) is projective (as \( R_R \)), there is an \( R \)-homomorphism \( h \) from \( R \) into \( P \) making

\[
\begin{array}{cccc}
  R & \xrightarrow{h} & P & \xrightarrow{f} & R/I & \rightarrow & 0 \\
\end{array}
\]

where \( \pi \) is the natural mapping, commutative. Since for any arbitrary \( p \in P \), \( f(p) = \pi(x) = fh(x) \) for some \( x \in R \), \( p - h(x) \in \text{Ker } f \). Hence

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$P = \text{Ker } f + h(R)$. Since the Ker $f$ is small, this implies that $P = h(R)$. Let $p_0 = h(1)$. Then $P = p_0R$ and $R^{p_0}p_0R \rightarrow 0$, where $t_{p_0}(x) = p_0x$ for all $x \in R$, is direct since $P$ is projective. Hence Ker $t_{p_0} = \{ r \in R \mid p_0r = 0 \}$ is a direct summand of $R$. Since $p_0 = h(1)$, Ker $h = \text{Ker } t_{p_0}$. If $h(I) = 0$, then Ker $t_{p_0} = I$ and $I$ is a direct summand of $R$. Hence there is a minimal right ideal $J$ in $R$ such that $R = J \oplus I$. Thus, by [2, p. 50], there is an idempotent $e \neq 0$ in $J$ such that $eI = 0$ is small. If $h(I) \neq 0$ then $h(I) \subset \text{Ker } f$ since $fh(I) = \pi(I) = 0$. Thus $h(I)$ is small. Since $h(R)$ is projective, there is an $R$-homomorphism $\phi$ from $h(R)$ making

$$
\begin{array}{ccc}
R & \xrightarrow{i} & h(R) \\
\downarrow & & \downarrow \phi \\
h & & 0
\end{array}
$$

where $i$ is the identity map, commutative. Since $h(I)$ is small, $\phi(h(I))$ is small in $R$ by [4, p. 93]. Let $\phi(p_0) = a \in R$. Then $p_0 = h(p_0) = h(a) = h(1)a = p_0a$. Therefore, $a = \phi(p_0) = \phi(p_0a) = a^2$ and $aI = \phi(h(I))$ is small. Clearly $a \neq 0$ since $h(p_0) = p_0$. Conversely, suppose there is a nonzero idempotent $e$ in $R$ such that $eI$ is small. Since $eI \in N$ by [1, Lemma 2.4], the right ideal $(I : e) = \{ r \in R \mid er \notin I \}$ is $I$. Define a mapping $g$ from $eR$ onto $R/I$ by $g(er) = r + I$ for all $er \in eR$. Since $er_1 = er_2$ then $r_1 - r_2 \in (I : e) = I$, $g$ is well defined and clearly $g$ is an $R$-homomorphism from $eR$ onto $R/I$. Furthermore since $eR$ is a direct summand of $R$, $eR$ is projective and since the kernel of $g$ is $eI$, which is small, $g$ is a projective cover for $R/I$.

**Lemma 2.** Let $I$ be a large maximal right ideal in $R$ and let $L = \{ x \in R \mid xI = 0 \}$. Then $L^2 = 0$.

**Proof.** If $x \neq 0, y \neq 0$ are elements in $L$ then $I \cap yR \neq 0$ and $x(ry) = 0$ for some $r \in R$ such that $yr \neq 0$ in $I$. If $xy \neq 0$, then $r \in I$ since the set $\{ r \in R \mid (xy)r = 0 \} = I$. This is impossible since $yr \neq 0$ and $y \in L$. Thus $L^2 = 0$.

**Proof of Theorem 1.** Suppose every irreducible $R$-module has a projective cover. Let $\overline{I}$ be a maximal right ideal of $R/N$. Then there is a maximal right ideal $I$ in $R$ such that $\overline{I} = I/N$. By Lemma 1, there is a nonzero idempotent $e$ in $R$ such that $eI$ is small. By [1, Lemma 2.4], $eI \subset N$. Since $e \in N$, $e + N$ is a nonzero left annihilator of $\overline{I}$. Hence by Lemma 2, $\overline{I}$ cannot be large. Since $\overline{I}$ is a maximal right ideal of $R/N$, $\overline{I}$ must be a direct summand of $R/N$ if $\overline{I}$ is not large. Thus by
[6, Lemma 3.1], $R/N$ must be a semisimple ring with the minimum condition for right ideals. Now let $x \in R$ such that $x^2 - x \in N$. If $x \in N$, by Zorn's Lemma, we can construct a right ideal $J^*$ in $R$ with the properties that $N \subseteq J^*$, $x \in J^*$ such that if $K$ is a right ideal which contains $J^*$ properly then $x \in K$. Then the right $R$-module $xR + J^*/J^*$ is irreducible and $(J^*: x) = \{ r \in R \mid xr \in J^* \}$ is a maximal right ideal of $R$. Hence there is an idempotent $e \neq 0$ in $R$ such that $e \cdot (J^*: x) \subseteq N$. Since $x^2 - x = x(x - 1) \in N$, $(x - 1) \in (J^*: x)$ and $e(x - 1) = ex - e \in N$.

Conversely, suppose $R$ is semiprimary and if $x + N$ is a nonzero idempotent in $R/N$ then there is a nonzero idempotent $e$ in $R$ such that $ex - e \in N$. If $I$ is a maximal right ideal of $R$, $I/N$ is a maximal right ideal of $R/N$, and since $R$ is semiprimary, there is a minimal right ideal $K/N$ in $R/N$ such that $K/N \cap I/N = N$ and $K/N \oplus I/N = R/N$ (see [4, p. 67]). Let $\tilde{x} = x + N$, for some $x \in R$, be a nonzero idempotent in $K/N$ such that $\tilde{x} \cdot (I/N) = N$. By hypothesis, there is a nonzero idempotent $e$ in $R$ such that $ex - e \in N$. Since $xI \subseteq N$ and $ex - e \in N$, $eI \subseteq N$. Thus by Lemma 1, $R/I$ has a projective cover.

The following corollary is related to Corollary 1 of [4, p. 76].

**Corollary.** A ring $R$ is local (i.e. $R/N$ is a division ring) if and only if $1$ is a primitive idempotent and every irreducible $R$-module has a projective cover.

**Proof.** If $R$ is a local ring then 1 and 0 are only idempotents in $R$, and since $N$ is the only maximal right (left) ideal in $R$, every irreducible $R$-module has a projective cover. Conversely, suppose every irreducible $R$-module has a projective cover and $1$ is a primitive idempotent in $R$. By Theorem 1, $R/N$ is a semisimple ring with the minimum condition on right ideals and if $x + N$ is a nonzero idempotent in $R/N$ there is a nonzero idempotent $e$ in $R$ such that $ex - e \in N$. Since $e1 \in N$ and $ex - e \in N$, $eI \subseteq N$. Thus by Lemma 1, $R/I$ has a projective cover.

**Proof of Theorem 2.** We only need to prove that if $R$ is commutative such that every irreducible $R$-module has a projective cover then idempotents modulo $N$ can be lifted. We first prove that if $x + N$ is an idempotent such that $(x + N)(R/N)$ is a minimal ideal in $R/N$ then $x - e \in N$ for some idempotent $e$ in $R$. Let $J^*$ be as in the proof of Theorem 1. Since $xR + N \supseteq J^* \supseteq N$ and $xR + N/N$ is a minimal ideal of $R/N$, $J^* = N$ since $J^*$ is properly contained in $xR + N$. As in the case of the proof of Theorem 1, there is an idempotent $e$ in $R$ such that $e \cdot (J^*: x) = e \cdot (N: x) \subseteq N$. Now $(N: ex) = (N: x) = (N: e)$ since $(N: x)$ is a maximal ideal and $(N: ex) \supseteq (N: x) \supseteq (N: e) \supseteq (N: ex)$. Thus
(1-e) ∈ (N: e) = (N: x) and x−xe ∈ N. Since ex−e ∈ N, this implies that x−e ∈ N. Now let g = g² in R such that xg ∈ N. Since e−x ∈ N, eg ∈ N. Let e′ = e − eg. Then g·e′ = 0 and e′·e′ = (e−eg)(e−eg) = e−eg − eg + eg = e′. e′+N = e + N = x + N. It is well known that if R/N is a semisimple ring with the minimum condition then 1+N = (x₁+N) + (x₂+N) + · · · + (xₙ+N) for some positive integer n where xᵢ−xᵢ² ∈ N, i = 1, 2, · · · , n, xᵢxⱼ ∈ N if i ≠ j and (N: xᵢ), for each i, is a maximal right ideal (see [2, p. 46 and p. 50]). By the above argument, we can choose an orthogonal set of idempotents e₁, e₂, · · · , eₙ in R such that eᵢ−eᵢ² ∈ N, i = 1, 2, · · · , n, and 1+N = (e₁+N) + (e₂+N) + · · · + (eₙ+N). Now let y+N be an arbitrary nonzero idempotent in R/N. Then y+N = (e₁y+N) + (e₂y+N) + · · · + (eₙy+N) and eᵢy·eⱼy ∈ N if i ≠ j and (N: eᵢy) is a maximal ideal for all i such that eᵢy ∈ N. There is an orthogonal set of idempotents a₁, a₂, · · · , aₙ in R such that y−(a₁+a₂+ · · · +aₙ)² ∈ N and (a₁+a₂+ · · · +aₙ)² = (a₁+a₂+ · · · +aₙ).

References


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