COMMUTATIVITY OF THE INVARIANT DIFFERENTIAL OPERATORS ON A SYMMETRIC SPACE

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It is known (see Helgason [1]) that the algebra of invariant differential operators on a Riemannian symmetric space is commutative. The algebra may be defined algebraically. We give here an algebraic proof of its commutativity.

Let \( g \) be a Lie algebra over a field of characteristic zero and let \( f \) be a subalgebra of \( g \). Extend the adjoint representation of \( f \) on \( g \) to the universal enveloping algebra \( U(g) \) of \( g \) so that \( \text{ad} \, x, \, x \in f \), acts as a derivation of \( U(g) \). Then \( (\text{ad} \, x)(u) = xu - ux \) for \( u \in U(g) \). For this is the case if \( u \) belongs to \( g \), and \( g \) generates \( U(g) \). It follows from the formula that the invariant elements of \( U(g) \)—those annihilated by the action of \( f \)—are the elements of \( U(g) \) which commute with the elements of \( f \). Let \( U(g)^f \) be the subalgebra of invariant elements.

Now let \( U(g)^f \) be the left ideal generated by \( f \) in \( U(g) \). This left ideal is preserved by the action of \( f \). Moreover, \( [U(g)^f]^f = U(g)^f \cap U(g)^f \) is a two-sided ideal in \( U(g)^f \). Let \( U(g, f) \) be the quotient algebra. If \( g \) is the Lie algebra of a Lie group \( G \) and \( f \) that of a connected subgroup \( K \) then \( U(g, f) \) is the algebra of invariant differential operators on the homogeneous space \( G/K \) (Smoke [2]). If \( K \) is compact and the fixed point set of an involutive automorphism of \( G \) then \( G/K \) is a Riemannian symmetric space and the algebra \( U(g, f) \) is commutative.

Theorem. Suppose that \( g \) is equipped with an involutive Lie algebra automorphism \( z \rightarrow \bar{z} \) and let \( f \) be the subalgebra of fixed elements. Suppose that \( f \) is reductive in \( g \). Then \( U(g, f) \) is commutative.

The proof essentially reduces to the fact than an automorphism and an antiautomorphism coincide on \( U(g, f) \).

Extend the Lie algebra automorphism \( z \rightarrow \bar{z} \) to an automorphism \( u \rightarrow \bar{u} \) of the algebra \( U(g) \). Then for \( x \in f \), \( (\text{ad} \, x)(\bar{u}) = xu - \bar{u}x = (xu) - (ux) = ((\text{ad} \, x)(u))^\ast \), so \( u \rightarrow \bar{u} \) preserves the subalgebra \( U(g)^f \). The ideal \( U(g)^f \) is also preserved, so \( [U(g)^f]^f \) is preserved and an automorphism is induced on \( U(g, f) \).

Recall that \( U(g) \) has a canonical antiautomorphism \( u \rightarrow u^\ast \), characterized by the fact that \( z^\ast = -z \) for \( z \in g \). We have \( (\text{ad} \, x)(u^\ast) \)

Received by the editors October 17, 1966.

1 This research was partly supported by the National Science Foundation, Grant GP-6392.
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\[ xu^* - u^*x = (xu - ux)^* = [(\text{ad} \ x)(u)]^* \] for \( x \in \mathfrak{f} \) and \( u \in U(\mathfrak{g}) \), so the subalgebra \( U(\mathfrak{g})^f \) is preserved. Evidently the left ideal \( U(\mathfrak{g})^f \) is mapped into the right ideal \( \mathfrak{f} U(\mathfrak{g}) \). Thus \( [U(\mathfrak{g})^f]^f \) is mapped into \( [\mathfrak{f} U(\mathfrak{g})]^f \). We must show that \( [\mathfrak{f} U(\mathfrak{g})]^f \) is contained in \( [U(\mathfrak{g})^f]^f \). Since \( \mathfrak{f} \) is reductive in \( \mathfrak{g} \) the representation of \( \mathfrak{f} \) on \( U(\mathfrak{g}) \) is semisimple and \( U(\mathfrak{g}) \) is the direct sum \( U(\mathfrak{g}) = U(\mathfrak{g})^f \oplus [\mathfrak{f}, U(\mathfrak{g})] \). Here, \([\mathfrak{f}, U(\mathfrak{g})]\) is the subspace generated by the commutators \( xu - ux \), \( x \in \mathfrak{f} \) and \( u \in U(\mathfrak{g}) \). But \( \mathfrak{f} U(\mathfrak{g}) \subset U(\mathfrak{g})^f + [\mathfrak{f}, U(\mathfrak{g})] \), and it follows that the projection of \( \mathfrak{f} U(\mathfrak{g}) \) on \( U(\mathfrak{g})^f \) is contained in the projection of \( U(\mathfrak{g})^f \). Since \( \mathfrak{f} U(\mathfrak{g}) \) and \( U(\mathfrak{g})^f \) are invariant subspaces of \( U(\mathfrak{g}) \) their projections on \( U(\mathfrak{g})^f \) are respectively their intersections with this subspace. This shows that \( [\mathfrak{f} U(\mathfrak{g})]^f \) is contained in \( [U(\mathfrak{g})^f]^f \). The antiautomorphism \( u \rightarrow u^* \) therefore preserves the ideal \([U(\mathfrak{g})^f]^f\), inducing an antiautomorphism on \( U(\mathfrak{g}, \mathfrak{f}) \).

It remains to show that the induced automorphism and antiautomorphism coincide. For this, we find a subspace of \( U(\mathfrak{g})^f \), mapping onto \( U(\mathfrak{g}, \mathfrak{f}) \), on which the automorphism and antiautomorphism of \( U(\mathfrak{g}) \) coincide.

Recall that \( U(\mathfrak{g}) \) is filtered by subspaces \( U_p(\mathfrak{g}) \), where \( U_p(\mathfrak{g}) \) is spanned by the monomials of degree at most \( p \) in the elements of \( \mathfrak{g} \). The associated graded algebra is the symmetric algebra \( S(\mathfrak{g}) \) of \( \mathfrak{g} \). If \( S^p(\mathfrak{g}) \) is the component of degree \( p \), the natural map from \( U_p(\mathfrak{g}) \) onto \( S^p(\mathfrak{g}) \) has kernel \( U_{p-1}(\mathfrak{g}) \) and takes a monomial \( z_1 \cdots z_p \) of degree \( p \) into the corresponding commutative monomial \( z_1 \cdots z_p \) in \( S^p(\mathfrak{g}) \). There is a linear map \( \lambda: S(\mathfrak{g}) \rightarrow U(\mathfrak{g}) \) defined by

\[
\lambda(z_1 \cdots z_p) = \frac{1}{p!} \sum_{\sigma} z_{\sigma 1} \cdots z_{\sigma p},
\]

the sum running over all permutations. It is clear that \( \lambda: S^p(\mathfrak{g}) \rightarrow U_p(\mathfrak{g}) \) followed by the natural map \( U_p(\mathfrak{g}) \rightarrow S^p(\mathfrak{g}) \) is the identity on \( S^p(\mathfrak{g}) \). This implies that the element \( z_1 \cdots z_p - \lambda(z_1 \cdots z_p) \) of \( U_p(\mathfrak{g}) \) belongs to \( U_{p-1}(\mathfrak{g}) \).

Now let \( \mathfrak{b} \) be the subspace of those elements \( y \) of \( \mathfrak{g} \) which satisfy \( y^* = -y \). Then \( \mathfrak{g} = \mathfrak{f} \oplus \mathfrak{b} \). We may regard \( S(\mathfrak{b}) \) as a subalgebra of \( S(\mathfrak{g}) \).

**Lemma.** \( U(\mathfrak{g})^f + \lambda [S(\mathfrak{b})] = U(\mathfrak{g}) \).

To prove the lemma, choose an ordered basis \((x_i)\) for \( \mathfrak{f} \) and ordered basis \((y_j)\) for \( \mathfrak{b} \). The Poincare-Birkhoff-Witt theorem then implies that the monomials \( y_{j_1} \cdots y_{j_p} x_{i_1} \cdots x_{i_q} \), \( j_1 \leq \cdots \leq j_p \) and \( i_1 \leq \cdots \leq i_q \), form a basis of \( U(\mathfrak{g}) \). Clearly \( U(\mathfrak{g})^f \) contains all but those of the form \( y_{j_1} \cdots y_{j_p} \). The ground field \( U_0(\mathfrak{g}) \) belongs to \( \lambda [S(\mathfrak{b})] \). Assume that \( U_{p-1}(\mathfrak{g}) \) is contained in \( U(\mathfrak{g})^f + \lambda [S(\mathfrak{b})] \). Since \( y_{j_1} \cdots y_{j_p} \)
$-\lambda(y_i, \cdots, y_p)$ belongs to $U_{p-1}(g)$ we find that $U_p(g)$ is contained in $U(g)\mathfrak{f} + \lambda[S(\mathfrak{b})]$, so the lemma follows inductively.

The action of $\mathfrak{f}$ on $g$ extends to $S(g)$, with $\mathrm{ad} \, x, \, x \in \mathfrak{f}$, again acting as a derivation. Since $\mathfrak{b}$ is invariant in $g$, $S(\mathfrak{b})$ is invariant in $S(g)$. It is easy to see that $\lambda : S(g) \to U(g)$ commutes with the action of $\mathfrak{f}$. It follows that $\lambda[S(\mathfrak{b})]$ is an invariant subspace of $U(g)$. Each of the spaces in the lemma decomposes as a direct sum under the action of $\mathfrak{f}$ and $U(\mathfrak{g})\mathfrak{f} = [U(\mathfrak{g})\mathfrak{f}]^\mathfrak{f} + \lambda[S(\mathfrak{b})]^\mathfrak{f}$. It follows that the natural map $U(\mathfrak{g})\mathfrak{f} \to U(g, \mathfrak{f})$ maps $\lambda[S(\mathfrak{b})]^\mathfrak{f}$ onto $U(g, \mathfrak{f})$.

It remains only to show that the automorphism and antiautomorphism of $U(\mathfrak{g})$ agree on $\lambda[S(\mathfrak{b})]^\mathfrak{f}$. In fact they agree on $\lambda[S(\mathfrak{b})]$. For if $y_1, \cdots, y_p$ are any elements of $\mathfrak{b}$ we have $\lambda(y_1 \cdots y_p)^* = (-1)^p \lambda(y_1 \cdots y_p)$. On the other hand, $\lambda(y_1 \cdots y_p)^* = (-1)^p \lambda(y_1 \cdots y_p)$ also, since $\lambda(y_1 \cdots y_p)$ is a sum over all permutations.

It is now clear that $U(g, \mathfrak{f})$ is commutative. If $v$ and $w$ belong to $U(g, \mathfrak{f})$ then $(vw)^* = (vw)^* = w^*v^* = \bar{w}^* = (\bar{w}v)^*$, so $vw = vw$.

References


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