COMMUTATIVITY OF THE INVARIANT DIFFERENTIAL OPERATORS ON A SYMMETRIC SPACE

WILLIAM SMOKE

It is known (see Helgason [1]) that the algebra of invariant differential operators on a Riemannian symmetric space is commutative. The algebra may be defined algebraically. We give here an algebraic proof of its commutativity.

Let \( g \) be a Lie algebra over a field of characteristic zero and let \( \mathfrak{f} \) be a subalgebra of \( g \). Extend the adjoint representation of \( \mathfrak{f} \) on \( g \) to the universal enveloping algebra \( U(g) \) of \( g \) so that \( \text{ad} \, x, \, x \in \mathfrak{f} \), acts as a derivation of \( U(g) \). Then \( (\text{ad} \, x) (u) = xu - ux \) for \( u \in U(g) \). For this is the case if \( u \) belongs to \( g \), and \( g \) generates \( U(g) \). It follows from the formula that the invariant elements of \( U(g) \)—those annihilated by the action of \( \mathfrak{f} \)—are the elements of \( U(g) \) which commute with the elements of \( \mathfrak{f} \). Let \( U(g)^\mathfrak{f} \) be the subalgebra of invariant elements.

Now let \( U(g)\mathfrak{f} \) be the left ideal generated by \( \mathfrak{f} \) in \( U(g) \). This left ideal is preserved by the action of \( \mathfrak{f} \). Moreover, \( [U(g)\mathfrak{f}]^\mathfrak{f} = U(g)\mathfrak{f} \cap U(g)^\mathfrak{f} \) is a two-sided ideal in \( U(g)\mathfrak{f} \). Let \( U(g, \mathfrak{f}) \) be the quotient algebra. If \( g \) is the Lie algebra of a Lie group \( G \) and \( \mathfrak{f} \) that of a connected subgroup \( K \) then \( U(g, \mathfrak{f}) \) is the algebra of invariant differential operators on the homogeneous space \( G/K \) (Smoke [2]). If \( G \) is compact and the fixed point set of an involutive automorphism of \( G \) then \( G/K \) is a Riemannian symmetric space and the algebra \( U(g, \mathfrak{f}) \) is commutative.

**Theorem.** Suppose that \( g \) is equipped with an involutive Lie algebra automorphism \( z \rightarrow z^\ast \) and let \( \mathfrak{f} \) be the subalgebra of fixed elements. Suppose that \( \mathfrak{f} \) is reductive in \( g \). Then \( U(g, \mathfrak{f}) \) is commutative.

The proof essentially reduces to the fact than an automorphism and an antiautomorphism coincide on \( U(g, \mathfrak{f}) \).

Extend the Lie algebra automorphism \( z \rightarrow z^\ast \) to an automorphism \( u \rightarrow \tilde{u} \) of the algebra \( U(g) \). Then for \( x \in \mathfrak{f} \), \( (\text{ad} \, x) (\tilde{u}) = xu - \tilde{u}x = (xu)^\ast - (ux)^\ast = ((\text{ad} \, x) (u))^\ast \), so \( u \rightarrow \tilde{u} \) preserves the subalgebra \( U(g)^\mathfrak{f} \). The ideal \( U(g)^\mathfrak{f} \) is also preserved, so \( [U(g)^\mathfrak{f}]^\mathfrak{f} \) is preserved and an automorphism is induced on \( U(g, \mathfrak{f}) \).

Recall that \( U(g) \) has a canonical antiautomorphism \( u \rightarrow u^\ast \), characterized by the fact that \( z^\ast = -z \) for \( z \in g \). We have \( (\text{ad} \, x) (u^\ast) \)

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\[= xu^* - u^*x = (xu - ux)^* = [(\text{ad} x)(u)]^* \] for \(x \in \mathfrak{f}\) and \(u \in U(\mathfrak{g})\), so the subalgebra \(U(\mathfrak{g})^\mathfrak{f}\) is preserved. Evidently the left ideal \(U(\mathfrak{g})\mathfrak{f}\) is mapped into the right ideal \(\mathfrak{f}U(\mathfrak{g})\). Thus \([U(\mathfrak{g})\mathfrak{f}]^\mathfrak{f}\) is mapped into \([\mathfrak{f}U(\mathfrak{g})]^\mathfrak{f}\). We must show that \([\mathfrak{f}U(\mathfrak{g})]^\mathfrak{f}\) is contained in \([U(\mathfrak{g})\mathfrak{f}]^\mathfrak{f}\). Since \(\mathfrak{f}\) is reductive in \(\mathfrak{g}\) the representation of \(\mathfrak{f}\) on \(U(\mathfrak{g})\) is semisimple and \(U(\mathfrak{g})\) is the direct sum \(U(\mathfrak{g}) = U(\mathfrak{g})^\mathfrak{f} \oplus \mathfrak{f}, U(\mathfrak{g})\). Here, \([\mathfrak{f}, U(\mathfrak{g})]\) is the subspace generated by the commutators \(xu - ux, x \in \mathfrak{f}\) and \(u \in U(\mathfrak{g})\). But \(\mathfrak{f}U(\mathfrak{g}) \subset U(\mathfrak{g})^\mathfrak{f} + [\mathfrak{f}, U(\mathfrak{g})]\), and it follows that the projection of \(\mathfrak{f}U(\mathfrak{g})\) on \(U(\mathfrak{g})^\mathfrak{f}\) is contained in the projection of \(U(\mathfrak{g})\mathfrak{f}\). Since \(\mathfrak{f}U(\mathfrak{g})\) and \(U(\mathfrak{g})\mathfrak{f}\) are invariant subspaces of \(U(\mathfrak{g})\) their projections on \(U(\mathfrak{g})^\mathfrak{f}\) are respectively their intersections with this subspace. This shows that \([\mathfrak{f}U(\mathfrak{g})]^\mathfrak{f}\) is contained in \([U(\mathfrak{g})\mathfrak{f}]^\mathfrak{f}\). The antiautomorphism \(u \mapsto u^*\) therefore preserves the ideal \([U(\mathfrak{g})\mathfrak{f}]^\mathfrak{f}\), inducing an antiautomorphism on \((U(\mathfrak{g}), \mathfrak{f})\).

It remains to show that the induced automorphism and antiautomorphism coincide. For this, we find a subspace of \(U(\mathfrak{g})^\mathfrak{f}\), mapping onto \((U(\mathfrak{g}), \mathfrak{f})\), on which the automorphism and antiautomorphism of \(U(\mathfrak{g})\) coincide.

Recall that \(U(\mathfrak{g})\) is filtered by subspaces \(U_p(\mathfrak{g})\), where \(U_p(\mathfrak{g})\) is spanned by the monomials of degree at most \(p\) in the elements of \(\mathfrak{g}\). The associated graded algebra is the symmetric algebra \(S(\mathfrak{g})\) of \(\mathfrak{g}\). If \(S_p(\mathfrak{g})\) is the component of degree \(p\), the natural map from \(U_p(\mathfrak{g})\) onto \(S_p(\mathfrak{g})\) has kernel \(U_{p-1}(\mathfrak{g})\) and takes a monomial \(z_1 \cdots z_p\) of degree \(p\) into the corresponding commutative monomial \(z_1 \cdots z_p\) in \(S_p(\mathfrak{g})\). There is a linear map \(\lambda: S(\mathfrak{g}) \rightarrow U(\mathfrak{g})\) defined by

\[\lambda(z_1 \cdots z_p) = \frac{1}{p!} \sum_{\sigma} z_{\sigma 1} \cdots z_{\sigma p},\]

the sum running over all permutations. It is clear that \(\lambda: S_p(\mathfrak{g}) \rightarrow U_p(\mathfrak{g})\) followed by the natural map \(U_p(\mathfrak{g}) \rightarrow S_p(\mathfrak{g})\) is the identity on \(S_p(\mathfrak{g})\). This implies that the element \(z_1 \cdots z_p - \lambda(z_1 \cdots z_p)\) of \(U_p(\mathfrak{g})\) belongs to \(U_{p-1}(\mathfrak{g})\).

Now let \(\mathfrak{b}\) be the subspace of those elements \(y\) of \(\mathfrak{g}\) which satisfy \(\bar{y} = -y\). Then \(\mathfrak{g} = \mathfrak{f} \oplus \mathfrak{b}\). We may regard \(S(\mathfrak{b})\) as a subalgebra of \(S(\mathfrak{g})\).

**Lemma.** \(U(\mathfrak{g})\mathfrak{f} + \lambda[S(\mathfrak{b})] = U(\mathfrak{g}).\)

To prove the lemma, choose an ordered basis \((x_i)\) for \(\mathfrak{f}\) and ordered basis \((y_j)\) for \(\mathfrak{b}\). The Poincare-Birkhoff-Witt theorem then implies that the monomials \(y_{j_1} \cdots y_{j_p} x_{i_1} \cdots x_{i_q}, j_1 \leq \cdots \leq j_p, i_1 \leq \cdots \leq i_q\), form a basis of \(U(\mathfrak{g})\). Clearly \(U(\mathfrak{g})\mathfrak{f}\) contains all but those of the form \(y_{j_1} \cdots y_{j_p}\). The ground field \(U_0(\mathfrak{g})\) belongs to \(\lambda[S(\mathfrak{b})]\). Assume that \(U_{p-1}(\mathfrak{g})\) is contained in \(U(\mathfrak{g})\mathfrak{f} + \lambda[S(\mathfrak{b})]\). Since \(y_{j_1} \cdots y_{j_p}\)
\(-\lambda(y_{j_1} \cdots y_{j_p})\) belongs to \(U_{p-1}(g)\) we find that \(U_p(g)\) is contained in \(U(g)\frac{1}{2} + \lambda[S(b)]\), so the lemma follows inductively.

The action of \(f\) on \(g\) extends to \(S(g)\), with \(\text{ad } x, x \in \mathfrak{f}\), again acting as a derivation. Since \(b\) is invariant in \(g\), \(S(b)\) is invariant in \(S(g)\). It is easy to see that \(\lambda: S(g) \to U(g)\) commutes with the action of \(f\). It follows that \(\lambda[S(b)]\) is an invariant subspace of \(U(g)\). Each of the spaces in the lemma decomposes as a direct sum under the action of \(f\) and \(U(g) = U(g)\frac{1}{2} + \lambda[S(b)]\). It follows that the natural map \(U(g)^f \to U(g, f)\) maps \(\lambda[S(b)]^f\) onto \(U(g, f)\).

It remains only to show that the automorphism and antiautomorphism of \(U(g)\) agree on \(\lambda[S(b)]\). In fact they agree on \(\lambda[S(b)]\). For if \(y_1, \cdots, y_p\) are any elements of \(b\) we have \(\lambda(y_1 \cdots y_p)^{-1} = (-1)^p \lambda(y_1 \cdots y_p)\). On the other hand, \(\lambda(y_1 \cdots y_p)^* = (-1)^p \lambda(y_1 \cdots y_p)\) also, since \(\lambda(y_1 \cdots y_p)\) is a sum over all permutations.

It is now clear that \(U(g, f)\) is commutative. If \(v\) and \(w\) belong to \(U(g, f)\) then \((vw)^{-1} = (vw)^* = w^* v^* = \bar{w} \bar{v} = (\bar{w} \bar{v})^*-1\), so \(vw = wv\).

References


University of California, Irvine