THE RELATION BETWEEN THE SEQUENCE-TO-SEQUENCE AND THE SERIES-TO-SERIES VERSIONS OF QUASI-HAUSDORFF SUMMABILITY METHODS

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1. Introduction. Let \((H, \mu_n)\) be a regular Hausdorff method of summability, and let

\[
(1) \quad t_n = \sum_{k=n}^{\infty} \binom{k}{n} (\Delta^{k-n} \mu_{n+1}) s_k,
\]

\[
(2) \quad b_n = \sum_{k=n}^{\infty} \binom{k}{n} (\Delta^{k-n} \mu_n) a_k,
\]

where \(s_k = a_0 + a_1 + \cdots + a_k\). We shall call \(A\) the summability method given by the sequence-to-sequence transformation (1), and \(B\) the summability method given by the series-to-series transformation (2).

It is proved in \([2]\) and \([3]\) that summabilities \(A\) and \(B\) are regular.

We shall say that the transformations (1) and (2) are equivalent if the convergence of (1) for all \(n\) implies the convergence of (2) for all \(n\), and conversely, and in either case, the sums are related by the equation

\[
(3) \quad t_n = b_0 + b_1 + \cdots + b_n.
\]

(1) may be written as

\[
t = H^*(\mu_{n+1}) s,
\]

where \(s, t\) denote the sequences \((s_k), (t_k)\), and \(H^*(\mu_{n+1})\) the matrix \((\alpha_{n,k})\), where

\[
\alpha_{n,k} = \binom{k}{n} (\Delta^{k-n} \mu_{n+1}) \quad (k \geq n),
\]

\[
= 0 \quad (k < n).
\]

We shall prove the following two theorems.

**Theorem 1.** If \(t_0 = b_0\), and

\[
(4) \quad H^*(\mu_{n+1}) \{ H^*(n+1) s \} = H^*(n+1) \{ H^*(\mu_{n+1}) s \},
\]

then the transformations (1) and (2) are equivalent.

**Theorem 2.** If, for all (fixed) \(n\),
as \( k \to \infty \), then the transformations (1) and (2) are equivalent.

2. **Proof of Theorem 1.** Let \( \bar{a} \) and \( \bar{b} \) denote the sequences \( \{ (n+1)a_{n+1} \} \) and \( \{ (n+1)(t_{n+1}-t_n) \} \). Then \( \bar{a} = -H^*(n+1)s \), and, by (4),

\[
\bar{b} = -H^*(n+1)t = -H^*(n+1) \left\{ H^*(\mu_{n+1})s \right\} \\
= -H^*(\mu_{n+1}) \left\{ H^*(n+1)s \right\} \\
= H^*(\mu_{n+1})\bar{a}.
\]

Hence

\[
(n + 1)(t_{n+1} - t_n) = \sum_{k=n}^{\infty} \binom{k}{n} (\Delta^{k-n} \mu_n)(k+1) a_{k+1}
\]

for \( n \geq 0 \). Noting that \( (k+1/n+1)C_{k,n} = C_{k+1,n+1} \) and replacing \( k+1 \) by \( k \) and \( n+1 \) by \( n \), we have

\[
t_n - t_{n-1} = \sum_{k=n}^{\infty} \binom{k}{n} (\Delta^{k-n} \mu_n) a_k = b_n
\]

for \( n \geq 1 \), and \( t_0 = b_0 \) by hypothesis. Thus (3) is satisfied, and the transformations (1) and (2) are equivalent.

3. **Proof of Theorem 3.** Write

\[
b_{n,K} = \sum_{k=n}^{K} \binom{k}{n} (\Delta^{k-n} \mu_n) a_k,
\]

\[
t_{n,K} = \sum_{k=n}^{K} \binom{k}{n} (\Delta^{k-n} \mu_{n+1}) s_k
\]

(both of these may be taken as 0 for \( n > K \)). If (5) holds, then, for any fixed \( n \), we have, as \( K \to \infty \)

\[
b_{n,K} = \sum_{k=n}^{K} \binom{k}{n} (\Delta^{k-n} \mu_n)(s_k - s_{k-1})
\]

\[
= \sum_{k=n-1}^{K} s_k \Delta \left\{ \binom{k}{n} (\Delta^{k-n} \mu_n) \right\} + o(1),
\]

where the \( \Delta \) outside the curly bracket is taken as operating on the variable \( k \), and the curly bracket is taken as 0 when \( k = n - 1 \). Now using
\[ \Delta^{k-n} \mu_n = \Delta^{k+1-n} \mu_n + \Delta^{k-n} \mu_{n+1} \]

we have

\[
\Delta \left\{ \binom{k}{n} \left( \Delta^{k-n} \mu_n \right) \right\} = \binom{k}{n} \left[ \Delta^{k+1-n} \mu_n + \Delta^{k-n} \mu_{n+1} \right] - \binom{k+1}{n} \Delta^{k+1-n} \mu_n
\]

(9)

\[ = -\binom{k}{n-1} \Delta^{k-(n-1)} \mu_n + \binom{k}{n} \Delta^{k-n} \mu_{n+1}, \]

where we take the second term on the right of (9) as meaning 0 in the case \( k = n - 1 \), and the first as meaning 0 when \( n = 0 \).

We deduce at once from (8) and (9) that, for fixed \( n \),

\[ t_{n,K} = b_{0,K} + b_{1,K} + \cdots + b_{n,K} + o(1) \]

as \( K \to \infty \), and this proves the theorem.

4. Examples. Now let us apply these ideas to some examples. We shall use the following lemma which is a paraphrase of Theorem 26 in [1].

**Lemma.** If, for any sequence \((p_k)\) which is monotonic decreasing for large enough \( k \), \( \sum_{k=n}^{\infty} a_k p_k \) exists, then

\[ \lim_{k \to \infty} p_k \sum_{l=n}^{k} a_l = 0. \]

(i) If \( \mu_n = \lambda^n \) \((0 < \lambda < 1)\), then (1) becomes

(10)

\[ t_n = \lambda^{n+1} \sum_{k=n}^{\infty} \binom{k}{n} (1 - \lambda)^{k-n} s_k. \]

This is the circle method of summation introduced by Hardy and Littlewood. (2) becomes

(11)

\[ b_n = \lambda^n \sum_{k=n}^{\infty} \binom{k}{n} (1 - \lambda)^{k-n} a_k, \]

and (5) becomes

(12)

\[ \binom{k}{n} (1 - \lambda)^{k-n} s_{k-1} \to 0. \]

The convergence of (10) for a given \( n \) implies (12) for that \( n \). Also, by the lemma quoted above with \( p_k = C_{k,n}(1 - \lambda)^{k-n} \), the convergence of (11) for a given \( n \) implies (12).
Since summability \( A \) asserts more than the convergence of (10) for all \( n \), and summability \( B \) asserts more than the convergence of (11) for each \( n \), we see at once that, in this case, summabilities \( A \) and \( B \) are equivalent.

(ii) If
\[
\mu_n = \binom{n + r}{r}^{-1},
\]
then (1) becomes
\[
t_n = r(n + 1) \sum_{k=n}^{\infty} \frac{k(k - 1)(k - 2) \cdots (k - n + 1)}{(k + r + 1)(k + r) \cdots (k + r - n)} s_k
\]
(13)
\[
= \frac{n + 1}{r + 1} \Delta^{-r} \left[ \frac{s_n}{\binom{n + r + 1}{n}} \right].
\]

This is the quasi-Cesáro transformation \((C^*, r)\) introduced by Kuttner [4]. (2) becomes
\[
b_n = r \sum_{k=n}^{\infty} \frac{k(k - 1) \cdots (k - n + 1)}{(k + r)(k + r - 1) \cdots (k + r - n)} a_k
\]
(14)
\[
= \Delta^{-r} \left[ \frac{a_n}{\binom{n + r}{n}} \right].
\]

For any given \( n \), the assertion that the series defining \( t_n \) converges is easily seen to be equivalent to
\[
\sum_{k=1}^{\infty} \frac{s_k}{k^2}
\]
(15)
converges, while the assertion that the series defining \( b_n \) converges is equivalent to
\[
\sum_{k=1}^{\infty} \frac{a_k}{k}
\]
(16)
converges. Condition (5) is easily seen to reduce, in the special case considered, to
\[
s_k = o(k).
\]
(17)

By the lemma quoted above with \( p_k = 1/k \), (16) implies (17). Hence, whatever \( r \), \( B \Rightarrow A \). On the other hand, it is clearly false that (15)
implies (17). But summability A asserts more than the convergence of (15), since (15) merely gives the existence of $t_n$. Thus this does not exclude the possibility that summability A might imply (17).

What we do, in fact, have is that $A \Rightarrow B$ is true when, $r \leq 1$, but not when $r > 1$. For recall that $A$ is $(C^*, r)$. It follows from the results of a paper by Kuttner [4] that $(C^*, r) \Rightarrow (C, 1)$ when $r \leq 1$; and it is well known that $(C, 1)$ implies (17). On the other hand, if $r > 1$, let $1 \leq \beta < \alpha < r$, and $s_k = (-1)^k(k+1)^\beta$. Then $(s_k)$ is summable $(C, \alpha)$, and hence summable $(C^*, r)$ [4]. But (17) is false. Indeed, (16) does not converge, so that $b_n$ is not defined.

References


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