LIMIT OF A SEQUENCE OF FUNCTIONS WITH ONLY COUNTABLY MANY POINTS OF DISCONTINUITY

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1. Introduction and statement of results.

1.1. Introduction. We present here first an approximation theorem (Theorem 1) for certain limit functions defined on a general topological space. This strengthens a result which may be found in Hausdorff [1]. With the aid of this theorem we characterize the limits of some classes of discontinuous functions in Theorem 2.

Denote by $S$ a topological space. All functions considered are real valued. Convergence means pointwise convergence unless otherwise stated. Suppose $f$ is a function defined on $S$, $x \in S$, and $f$ is not continuous at $x$. The statement that $(x, f(x))$ is a removable point of discontinuity means that there exists a function $g$ which agrees with $f$ on $S - \{x\}$ and which is continuous at $x$. The statement that the function $u$ defined on $S$ is upper semicontinuous means that, if $x \in S$ and $d > u(x)$, then there exists a neighborhood $V$ of $x$ such that, if $y \in V$, then $d > u(y)$. The function $l$ is lower semicontinuous if $-l$ is upper semicontinuous.

1.2. Statement of Theorems.

Theorem 1. Suppose $M$ is a linear space of real valued functions defined on $S$ which contains a nonzero constant function and which is closed under the operation of absolute value, and $U$ is the set to which $u$ belongs only in case $u$ is the greatest lower bound of a countable subset of $M$. Then, if the function $f$ defined on $S$ is the limit of a sequence of functions in $M$, it is the uniform limit of a sequence each term of which is the difference of two members of $U$, each of which is bounded above.

Theorem 2. Suppose $S$ is perfectly normal and $f$ is a function defined on $S$. Each two of the following three statements are equivalent:

1. the function $f$ is the limit of a sequence of functions, each of which has at most a finite number of points of discontinuity, each of which is removable;

2. the function $f$ is the limit of a sequence of functions, each of which has at most countably many points of discontinuity; and

3. there exist a function $g$ which is the limit of a sequence of continuous functions and a countable subset $T$ of $S$ such that, if $x \in S - T$, $f(x) = g(x)$.

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2. Proof of Theorems.

2.1. Proof of Theorem 1. The following facts should be noted: As \( M \) is closed under the operation of absolute value, if each of \( h \) and \( k \) is in \( M \) then each of \( \max \{h, k\} \) and \( \min \{h, k\} \) is in \( M \). Also, each member of \( U \) is the limit from above of a monotonic sequence of members of \( M \).

Suppose \( \{f_p\}_{p=1}^\infty, f_p \in M, \) converges to \( f \). For each positive integer \( p \) define \( l_p = \text{l.u.b.} \{f_p, f_{p+1}, \ldots \} \) and \( u_p = \text{g.l.b.} \{f_p, f_{p+1}, \ldots \} \). Thus \( -l_p \in U, u_p \in U, \) \( l_p \geq f_p \geq u_p, \) \( l_{p+1} \leq l_p, \) and \( u_p \leq u_{p+1} \). Suppose \( c > 0 \). Define \( R_p = \{x \mid l_p(x) - u_p(x) \leq c\} \). We assume that \( R_1 \) contains at least one point. Note that \( \bigcup_{p=1}^\infty R_p \neq \emptyset \).

Now we show that the function \( g_p \) defined such that \( g_p(x) = 1 \) if \( x \in R_p \) and \( g_p(x) = 0 \) if \( x \notin R_p \) is in \( U \). There exists a monotonic sequence \( \{v_n\}_{n=1}^\infty, v_n \in M, \) converging from above to \( u_p \) and there exists a monotonic sequence \( \{w_n\}_{n=1}^\infty, w_n \in M, \) converging from below to \( l_p \). Define \( h_n = w_n - v_n \) and \( Q_n = \{x \mid h_n(x) \leq c\} \). Thus \( R_p = \bigcap_{n=1}^\infty Q_n \).

Define \( q_n(x) = 1 \) if \( x \in Q_n \) and \( q_n(x) = 0 \) if \( x \notin Q_n \). Define \( d_n = \max \{1 - (\max \{h_n, c\} - c), 0\} \). Now \( d_n(x) = 1 \) if \( x \in Q_n \) and \( 0 \leq d_n(x) < 1 \) if \( x \notin Q_n \). Define \( r_{n,i} = \max \{i \cdot d_n - i + 1, 0\} \). Thus \( \{r_{n,i}\}_{i=1}^\infty, r_{n,i} \in M, \) is a monotonic sequence converging to \( q_n \) from above. Define \( \alpha_{i,p} = \min\{r_{1,i}, r_{2,i}, \ldots, r_{i,i}\} \). Then it is true that \( \{\alpha_{i,p}\}_{i=1}^\infty, \alpha_{i,p} \in M, \) is a monotonic sequence converging to \( g_p \) from above and therefore \( g_p \in U \).

Define \( f_0 = 0, \)
\[
s_p = \sum_{n=1}^p \max\{f_n - f_{n-1}, 0\},
\]
and
\[
t_p = \sum_{n=1}^p \min\{f_n - f_{n-1}, 0\}.
\]

Note that \( s_p + t_p = f_p \). Define \( h(x) = s_p(x) \) and \( k(x) = t_p(x) \) if \( x \in R_p \) but \( x \notin R_{p-1} \). Note that both \( k \) and \( -h \) are bounded above.

Now we show that each of \( k \) and \( -h \) is in \( U \). Define \( \beta_{i,p} = i \cdot \alpha_{i,p} - i \). Thus \( \beta_{i,p}(x) = 0 \) for \( x \in R_p, \) \( \beta_{i,p}(x) < 0 \) for \( x \notin R_p, \) and \( \beta_{i,p}(x) \to -\infty \) as \( i \to \infty \) if \( x \in R_p \). Define \( \delta_{i,1} = \max\{\beta_{i,1} + l_1, l_2\} \). Define \( \delta_{i,n} = \max\{\beta_{i,n} + \delta_{i,n-1}, t_{n+1}\} \) if \( n > 1 \). Define \( \gamma_i = \delta_{i,i} \). If \( p \leq i \) and \( x \in R_p, \)
\[
\gamma_i(x) = \delta_{i,p}\) is a monotonic sequence converging from above to \( k \) and \( k \in U \). A similar argument shows that \( -h \) is in \( U \).

If \( x \in R_p \) but \( x \notin R_{p-1} \), then \( h(x) = s_p(x), \) \( k(x) = t_p(x), \) and \( h(x) + k(x) = f_p(x) \). By the way \( R_p \) was defined.
\[ |f(x) - h(x) - k(x)| = |f(x) - f_p(x)| \leq c. \]

2.2. Notation. Suppose \( R \) is a subset of \( S \). Denote by \( U(R) \), \( L(R) \), and \( C(R) \), respectively, the set of all upper semicontinuous functions, lower semicontinuous functions, and continuous functions defined on \( R \). Denote by \( C_1(R) \) the set of all functions which are the limit of a sequence of members of \( C(R) \). Denote by \( U(R) + L(R) \) the set \( \{ f \mid f = h + k, h \in U(R), k \in L(R) \} \).

2.3. Some properties of semicontinuous functions. Certain facts concerning semicontinuous functions should be recalled. Property (1) is that if \( \{ f_p \}_{p=1}^\infty, f_p \in C(S), \) converges to \( f \), then there exists a sequence \( \{ u_p \}_{p=1}^\infty, u_p \in U(S), u_{p+1} \geq u_p, \) converging to \( f \) and a sequence \( \{ l_p \}_{p=1}^\infty, l_p \in L(S), l_{p+1} \leq l_p, \) converging to \( f \) such that \( l_p \geq f_p \geq u_p \). This can be verified by defining \( u_p = \text{g.l.b.} \{ f_n \mid n = p, p + 1, \ldots \} \) and \( l_p = \text{l.u.b.} \{ f_n \mid n = p, p + 1, \ldots \} \). Property (2) is that \( S \) is perfectly normal if and only if every \( u \in U(S) \) is the limit from above of a monotonic sequence of continuous functions. This is a theorem due to Hing Tong [2].

2.4 Proof of Theorem 2.

(a) 1\( \rightarrow \)2: This follows immediately.

(b) 2\( \rightarrow \)3: Suppose \( \{ f_p \}_{p=1}^\infty \) is a sequence of functions defined on \( S \) converging to \( f \) such that each term of the sequence has at most countably many points of discontinuity. Define \( T = \{ x \mid f_p \) is discontinuous at \( x \) for some positive integer \( p \} \). The set \( T \) is countable. Define \( R = S - T, \) \( r_p \) to be the restriction of \( f_p \) to \( R \), and \( r \) to be the restriction of \( f \) to \( R \). Thus, \( \{ r_p \}_{p=1}^\infty, r_p \in C(R), \) converges to \( r \). Now we apply Theorem 1. We take \( M \) to be the set of all continuous functions defined on \( R \). The limit from above of a monotonic sequence of continuous functions is upper semicontinuous. Therefore, the set \( U \) of Theorem 1 is a subset of \( U(R) \). Thus, \( r \) is the uniform limit of a sequence \( \{ b_p \}_{p=1}^\infty \), where each \( b_p \) is the difference of two upper semicontinuous functions, each bounded above. A function \( u \in U(R) \) which is bounded above can be extended to a function \( v \in U(S) \) by defining \( v(y) = \text{g.l.b.} \{ u(y) \mid y \in V \} \) \( V \) a neighborhood of \( x \) where this lower bound exists. There exist at most a countable number of points \( x_1, x_2, \ldots \) of \( S \) where the lower bound does not exist. At these points define \( v(x_p) = -p \). By this means \( b_p \) can be extended to a function \( d_p \in U(S) + L(S) \). Since \( \{ d_p \}_{p=1}^\infty \) converges uniformly to \( f \) on \( R \), we can assume without loss of generality that \( |d_p(x) - f(x)| < 1/p \) for \( x \in R \) and all \( p \). Define \( c_p = d_p + 1/p \) and \( e_p = d_p - 1/p \). If \( x \in R \) and \( p \) and \( j \) are positive integers, \( c_p(x) \geq f(x) \geq e_j(x) \). Also, if \( x \in R \), \( c_p(x) \rightarrow f(x) \) as \( p \rightarrow \infty \) and \( e_p(x) \rightarrow f(x) \) as \( p \rightarrow \infty \).
Define $s_p = \min \{ c_1, c_2, \ldots, c_p \}$ and $t_p = \max \{ e_1, e_2, \ldots, e_p \}$. If $x \in R$, $s_p(x) \leq f(x) \leq t_p(x)$. If $x \in S$, $s_p(x) \leq s_{p+1}(x)$ and $t_p(x) \leq t_{p+1}(x)$.

Define $v_p = \min \{ \max \{ s_1, t_1 \}, \max \{ s_2, t_2 \}, \ldots, \max \{ s_p, t_p \} \}$ and $w_p = \max \{ \min \{ s_1, t_1 \}, \min \{ s_2, t_2 \}, \ldots, \min \{ s_p, t_p \} \}$. If $x \in R$, $v_p(x) \to f(x)$ as $p \to \infty$ and $w_p(x) \to f(x)$ as $p \to \infty$. If $x \in S$, $v_p(x) \leq v_{p+1}(x)$ and $w_p(x) \geq w_{p+1}(x)$.

If both $\alpha$ and $\beta$ are in $C_1(S)$ then both $\max \{ \alpha, \beta \}$ and $\min \{ \alpha, \beta \}$ are also in $C_1(S)$. From this and property (2) of §2.3 it follows that $v_p$ and $w_p$ are each in $C_1(S)$.

By property (1) of §2.3, for each positive integer $p$, there exist a sequence $\{ l_{p,n} \}_{n=1}^{\infty}$, $l_{p,n} \in L(S)$, $l_{p,n} \geq l_{p,n+1}$, converging to $v_p$ on $S$ and a sequence $\{ u_{p,n} \}_{n=1}^{\infty}$, $u_{p,n} \in U(S)$, $u_{p,n} \leq u_{p,n+1}$, converging to $w_p$ on $S$. Define $m_p = \min \{ l_1, l_2, \ldots, l_p \}$ and $q_p = \max \{ u_1, u_2, \ldots, u_p \}$. The sequence $\{ m_p \}_{p=1}^{\infty}$, $m_p \in L(S)$, $m_p \geq m_{p+1}$, converges to $f$ on $R$. The sequence $\{ q_p \}_{p=1}^{\infty}$, $q_p \in U(S)$, $q_p \leq q_{p+1}$, converges to $f$ on $R$. Further, $m_p \geq q_p$.

Define $g$ to be the average of the limit of $\{ m_p \}_{p=1}^{\infty}$ and the limit of $\{ q_p \}_{p=1}^{\infty}$. As both sequences converge to $f$ on $R$, $g$ agrees with $f$ on $R$. Define $\alpha_p = m_p + 1/p$ and $\beta_p = q_p - 1/p$. As $\alpha_p \in L(S)$, $\beta_p \in U(S)$, and $\alpha_p > \beta_p$, by a theorem due to Nagami [3] there exists a function $g_p \in C(S)$ such that $\alpha_p > g_p > \beta_p$. Denote the points of $T$ as $\{ x_1, x_2, \ldots \}$. The function $g_p$ can be chosen so that it agrees with $g$ at the first $p$ points of $T$. This statement can be justified as follows: Consider the first two points of $T$, $x_1$ and $x_2$. If every neighborhood of $x_1$ contains $x_2$ or vice versa then every continuous function has the same value at $x_1$ and $x_2$. This is also true of the functions in $C_1(S)$, in particular those in $U(S)$ and $L(S)$. Thus, $g(x_1) = g(x_2)$. As $S$ is perfectly normal, $x_1$ and $x_2$ are contained in a closed subset of a neighborhood $V$ of $x_1$ or $x_2$ which has the property that there exist numbers $a_1$ and $a_2$ such that if $z \in V$, $\alpha_p(z) > a_1 > g(x_1) > a_2 > \beta_p(z)$. If there is a neighborhood of $x_1$ which does not contain $x_2$ and vice versa, then there exist a neighborhood $V$ of $x_1$ and a neighborhood $W$ of $x_2$ such that $x_1$ belongs to $V$ which does not intersect $W$ and there exist numbers $a_1$ and $a_2$ such that if $z \in V$, $\alpha_p(z) > a_1 > g(x_1) > a_2 > \beta_p(z)$. The neighborhood $W$ has similar properties. In either case Urysohn’s lemma can be used to modify $g_p$ such that it agrees with $g$ at $x_1$ and $x_2$ but is still continuous and between $\alpha_p$ and $\beta_p$. This process can be continued for a finite number of points of $T$.

The sequence $\{ g_p \}_{p=1}^{\infty}$ converges to $g$. If $x \in R$, $f(x) = g(x)$.

(c) Suppose $\{ g_p \}_{p=1}^{\infty}$, $g_p \in C(S)$, converges to $g$. Denote the members of $T$ as $\{ x_1, x_2, \ldots \}$. For each positive integer $p$ define
\[ f_p(x) = f(x) \quad \text{if } x = x_n, \quad n \leq p, \]
\[ = g_p(x) \quad \text{if } x \neq x_n, \quad n \leq p, \]
for \( x \in S \). The function \( f_p \) has at most a finite number of discontinuities, each of which is removable. The sequence \( \{f_p\}_{p=1}^{\infty} \) converges to \( f \).

**References**


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