INVERSE LIMITS OF PERFECTLY NORMAL SPACES

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It is the purpose of this note to establish that the countable inverse limit of perfectly normal topological spaces is again a perfectly normal space and to establish some of the consequences of that fact as it pertains to Moore spaces. In particular, this result yields a new proof of the fact that, if there is a normal nonmetrizable (complete) Moore space, then there is one which is not locally metrizable at any point [2], [6].

Throughout this note, Cl(M) denotes the closure of the point set M. If \( \{X_n, \pi^n_m\} \) is an inverse mapping system with inverse limit \( X_\infty \) and \( n \) is a positive integer, \( \pi_n \) denotes the projection of \( X_\infty \) into \( X_n \) (all inverse mapping systems are taken over the set of positive integers directed by \(<\)).

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**Theorem 1.** If \( X_\infty \) is the inverse limit of the inverse mapping system \( \{X_n, \pi^n_m\} \) where, for each \( n \), \( X_n \) is a topological space in which each closed set is an inner limiting (= Gδ) set and \( \pi^n_{n+1} \) is a mapping of \( X_{n+1} \) into \( X_n \), then each closed subset of \( X_\infty \) is an inner limiting set.

**Proof.** Suppose that \( \{X_n, \pi^n_m\} \) and \( X_\infty \) are as in the hypothesis and \( M \) is a closed subset of \( X_\infty \). There exist sequences \( 0_{11}, 0_{12}, \cdots; 0_{21}, 0_{22}, \cdots; \cdots \) such that, for each integer \( n \), \( 0_{n1}, 0_{n2}, \cdots \) is a sequence of domains whose common part is Cl(\( \pi_n(M) \)) and, for each positive integer \( m \), \( 0_{nm} \) contains \( 0_{n,m+1} \) and \( (\pi^n_{n+1})^{-1}(0_{nm}) \) contains \( 0_{n+1,m} \). Clearly, for each \( n \), \( \pi_n^{-1}(0_{nn}) \) contains \( M \). Suppose that \( p \) is a point of \( X_\infty \) not in \( M \) such that, for each \( n \), \( \pi_n^{-1}(0_{nn}) \) contains \( p \). For some positive integer \( n \), \( \pi_n(p) \) is not in Cl(\( \pi_n(M) \)) and, hence, for some \( m > n \), \( \pi_n(p) \) is not in \( 0_{nm} \). Then \( \pi_m(p) \) is not in \( 0_{mm} \), which is a subset of \( (\pi^n_m)^{-1}(0_{nm}) \). Thus \( p \) is not in \( \pi^{-1}(0_{mm}) \) and \( M \) is an inner limiting set.

**Theorem 2.** If \( X_\infty \) is the inverse limit of the inverse mapping system \( \{X_n, \pi^n_m\} \) where, for each \( n \), \( X_n \) is a perfectly normal space and \( \pi^n_{n+1} \) is a mapping of \( X_{n+1} \) into \( X_n \), then \( X_\infty \) is perfectly normal.

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Proof. Let \( \{X_n, \pi^n\} \) and \( X_\infty \) be as in the hypothesis and let \( H \) and \( K \) be mutually exclusive closed point sets in \( X_\infty \). It follows from Theorem 1 that every closed set in \( X_\infty \) is an inner limiting set.

There exist sequences \( 0_{11}, 0_{12}, \ldots ; 0_{21}, 0_{22}, \ldots ; \ldots \) such that, for each \( n \), \( 0_{n1}, 0_{n2}, \ldots \) is a sequence of domains in \( X_n \) whose common part is \( \text{Cl}(\pi_n(H)) \cdot \text{Cl}(\pi_n(K)) \) and, for each \( m \), \( 0_{nm} \) contains \( \text{Cl}(0_{nm+1}) \) and \( (\pi^{n+1}_n)^{-1}(0_{nm}) \) contains \( \text{Cl}(0_{n+1,m}) \). Now, \( \text{Cl}(\pi_1(H)) \setminus 0_{11} \) and \( \text{Cl}(\pi_1(K)) \setminus 0_{11} \) are two mutually exclusive closed point sets in \( X_1 \), the former does not intersect \( \text{Cl}(\pi_1(K)) \), and the latter does not intersect \( \text{Cl}(\pi_1(H)) \). Therefore, there exist in \( X_1 \) domains \( D_1 \) and \( E_1 \) containing \( \text{Cl}(\pi_1(H)) \setminus 0_{11} \) and \( \text{Cl}(\pi_1(K)) \setminus 0_{11} \) respectively such that \( \text{Cl}(D_1) \) does not intersect \( \text{Cl}(E_1) \) or \( \text{Cl}(\pi_1(K)) \) and \( \text{Cl}(E_1) \) does not intersect \( \text{Cl}(\pi_1(H)) \). Similarly, there exist in \( X_2 \) domains \( D_2 \) and \( E_2 \) containing \( (\pi^2_2)^{-1}(\text{Cl}(D_1)) + \text{Cl}(\pi_2(H)) \setminus 0_{22} \) and \( (\pi^2_1)^{-1}(\text{Cl}(E_1)) + \text{Cl}(\pi_2(K)) \setminus 0_{22} \) respectively such that \( \text{Cl}(D_2) \) does not intersect \( \text{Cl}(E_2) \) or \( (\pi^2_2)^{-1}(\text{Cl}(E_1)) + \text{Cl}(\pi_2(K)) \) and \( \text{Cl}(E_2) \) does not intersect \( (\pi^2_2)^{-1}(\text{Cl}(D_1)) + \text{Cl}(\pi_2(H)) \). This process may be continued to obtain sequences \( D_1, D_2, D_3, \ldots \) and \( E_1, E_2, E_3, \ldots \) such that, for each positive integer \( n \),

(i) \( D_n \) and \( E_n \) are mutually exclusive domains in \( X_n \);

(ii) \( \pi^{-1}_n(D_n) \) is a subset of \( \pi^{-1}_{n+1}(D_{n+1}) \) and \( \pi^{-1}_n(E_n) \) is a subset of \( \pi^{-1}_{n+1}(E_{n+1}) \); and

(iii) \( D_n \) contains \( \text{Cl}(\pi_n(H)) \setminus 0_{nn} \) and \( E_n \) contains \( \text{Cl}(\pi_n(K)) \setminus 0_{nn} \).

Let \( D = \pi^{-1}_1(D_1) + \pi^{-1}_2(D_2) + \cdots \) and \( E = \pi^{-1}_1(E_1) + \pi^{-1}_2(E_2) + \cdots \).

It follows from (i) and (ii) that \( D \) and \( E \) are mutually exclusive domains in \( X_\infty \). It remains to be proved that \( D \) contains \( H \) and \( E \) contains \( K \).

Suppose that \( p \) is a point of \( H \). Then \( p \) is not in \( K \) and there exists a positive integer \( n \) such that \( \pi_n(p) \) is not in \( \text{Cl}(\pi_n(K)) \). There is a positive integer \( m > n \) such that \( \pi_m(p) \) is not in \( 0_{nm} \). Now, \( \pi_m(p) \) is not in \( (\pi^m_n)^{-1}(0_{nm}) \) and is, therefore, not in \( 0_{mm} \). Thus \( \pi_m(p) \) is in \( D_m \) and, hence, \( p \) is in \( \pi^{-1}_m(D_m) \). Similarly, \( E \) contains \( K \).

The following Theorem of Katetov [3] is a corollary of Theorem 2.

**Theorem of Katetov.** If all spaces \( P_1 \times \cdots \times P_n \) \((n = 1, 2, \cdots)\) are perfectly normal, then the space \( P = P_1 \times P_2 \times \cdots \) is perfectly normal as well.

Proof. For each \( n \), let \( \pi^{n+1}_n \) be the projection of \( P_1 \times \cdots \times P_{n+1} \) onto \( P_1 \times \cdots \times P_n \). Then the inverse limit of the inverse mapping system so obtained is topologically equivalent to \( P \) and is perfectly normal.

**Remark.** Katetov has shown, [3], by an example that his above
stated theorem does not hold if perfectly normal is replaced by completely normal. Therefore, our Theorem 2 does not hold if perfectly normal is replaced by completely normal even if all of the bonding mappings are onto.

**Theorem 3.** If \( X_\infty \) is the inverse limit of the inverse mapping system \( \{ X_n, \pi_{n+1} \} \) where, for each \( n \), \( X_n \) is a (complete) normal Moore space and \( \pi_{n+1} \) is a mapping of \( X_{n+1} \) into \( X_n \), then \( X_\infty \) is a (complete) normal Moore space.

**Proof.** It is known (see, for example, [1] and [7]) that the Cartesian product of countably many (complete) Moore spaces is a (complete) Moore space. Every subspace of a Moore space is a Moore space and every closed subspace of a complete Moore space is complete, [5]. Thus, the countable inverse limit of Moore spaces is a Moore space and, since the inverse limit is closed in the Cartesian product space [4, p. 31], it is complete if the coordinate spaces are complete. Now, each normal Moore space is perfectly normal and the Theorem is proved.

**Corollary 1.** If each Cartesian product of two normal Moore spaces is normal, then so is each Cartesian product of countably infinitely many.

The proof of this Corollary is precisely the same as that for the above Theorem of Katetov, or, as has been pointed out by Bruce Anderson, [1], it may be noted that this corollary follows from the Theorem of Katetov.

D. R. Traylor and the second author of this paper showed [2] that, if \( S^0 \) is a Moore space, there is a Moore space \( S^\omega \) such that every open set in \( S^\omega \) contains a copy of \( S^0 \) and \( S^\omega \) is normal if \( S^0 \) is. Traylor showed [6] that \( S^\omega \) is completable in such a way that the completion of \( S^\omega \) is normal if \( S^0 \) is complete and normal. The proofs of these results are simplified considerably by using Theorem 3 in the following:

**Corollary 2.** If \( S^0 \) is a Moore space then there is a Moore space \( S_\infty \) in which every open set contains a topological copy of \( S^0 \) and \( S_\infty \) is complete or normal, respectively, if \( S^0 \) is complete or normal.

**Proof.** Let \( S^0 \) be a Moore space. The notation of [2] will be used. Let \( M^j, S_{p,k}^j \), and \( S^j \) be as in [2], i.e., (1) \( M^j \) is a dense subset of \( S^j \), (2) \( S_{p,1}^{j+1}, S_{p,2}^{j+1}, \ldots \) is a null sequence, of copies of \( S^0 \), which converges in \( S_j^{j+1} \) to the point \( P \) of \( M^j \), and (3) \( S_j^{j+1} = S^j + \sum_{p \in M_j} \sum_{k=1}^{n_p} S_{p,k}^{j+1} \). Define \( \pi_j^{j+1} \) by \( \pi_j^{j+1}(x) = x \) if \( x \) is in \( S^j \) and \( \pi_j^{j+1}(x) = P \) if there is a positive integer \( k \) such that \( x \) is in \( S_{p,k}^{j+1} \). Then \( \pi_j^{j+1} \) is a continuous.
transformation of $S^{i+1}$ onto $S^i$. The inverse limit, $S_\infty$, of the inverse mapping system \( \{ S^n, \pi^n_m \} \) is the desired space.

\section*{References}


