THE OSGOOD-TAYLOR-CARATHÉODORY THEOREM

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1. **Introduction.** In 1903 W. F. Osgood made the now famous conjecture that if Ω_1 is a region bounded by a Jordan curve and χ a function mapping Ω_1 conformally onto another such region Ω_2, then χ can be extended to a homeomorphism of $\bar{\Omega}_1$ onto $\bar{\Omega}_2$. Although Osgood managed to establish special cases of his conjecture in the interim, it was not until 1913 that the assertion was fully validated by Osgood and Taylor [**7**] and independently by Carathéodory [**3**]. We shall refer to this result as the Osgood-Taylor-Carathéodory theorem. The methods of proof employed in the two papers were quite different, but each was rather technical. Simplified derivations have since been given, notably that by R. Courant [**4**, pp. 400–405].

In reexamining these questions of boundary extensions, we propose an approach that adheres to the basic viewpoint of Courant but makes three essential modifications. The first modification is to replace the Jordan region Ω_1 by the open unit disc ω. This might appear offhand to be disadvantageous, since, in the original situation, one would only have to show that χ can be extended to a continuous mapping of $\bar{\Omega}_1$ onto $\bar{\Omega}_2$ and then apply symmetry to conclude the corresponding property for χ^{-1}. In fact, however, it turns out to be a definite advantage. Not only does the choice of $\Omega_1 = \omega$ give rise to evident simplifications, but also it serves to bear out an intrinsic lack of symmetry, leading to the consideration of more general regions Ω_2. Moreover, the symmetrical case is very easily dealt with by other means. The second modification consists of allowing Ω_2 to be more general in the following specific way: Ω_2 can be taken as any region bounded by a closed curve (not necessarily simple). The third modification concerns the handling of topological properties of the boundary. In Courant's discussion emphasis is placed on the property of accessibility of boundary points. However, easy examples reveal that this is not enough to ensure that χ can be continuously extended to $\bar{\omega}$ (see the discussion centered about Figure 1). Local accessibility would do, but an even better property is that of local sequential accessibility, defined in §2.

Our scheme is to formulate the extension theorems first in a general topological setting, so that the key ideas are brought sharply into

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focus, and then to show how these theorems specialize to the case of conformal and quasiconformal mappings. It should be remarked that some of the techniques used here are further developed in an analytical context in [1] and in applications to topological problems in [2].

2. **The topological extension theorems.** Here we start with a bounded plane region Ω and a homeomorphism χ of the open unit disc ω onto Ω. The problem is to extend χ continuously to $\bar{\omega}$, and we note at the outset that if such an extension exists, it is unique and maps $\partial\omega$ onto $\partial\Omega$. For the existence of such an extension it is necessary and sufficient that χ have a limit at each point z_0 of $\partial\omega$ or, equivalently, that the limiting oscillation of χ vanish at each such z_0. We proceed to show that a much weaker oscillation condition will suffice provided $\partial\Omega$ is assumed to be parametrizable as a closed curve.

In what follows, the neighborhood of radius r about z will be designated as $N_r(z)$ and its circumference as $C_r(z)$. The oscillation of χ with which we shall work is that taken over the arc $\omega\cap C_r(z_0)$. This will be written as $\sigma_r(z_0)$, so that

$$\sigma_r(z_0) = \sup|\chi(z) - \chi(z')| \quad \text{for } z, z' \text{ on } \omega \cap C_r(z_0).$$

A point ζ_0 of $\partial\Omega$ will be called *locally sequentially accessible* if, for each sequence $\{\zeta_n\}$ of points of Ω converging to ζ_0 and each $\rho>0$, the open set $\Omega\cap N_\rho(\zeta_0)$ has a component containing infinitely many ζ_n.

LEMMA. *If Ω is a bounded plane region for which $\partial\Omega$ is parametrizable as a closed curve, then all points of $\partial\Omega$ are locally sequentially accessible.*

PROOF. Suppose the conclusion false, so that for some sequence $\{\zeta_n\}$ of points of Ω converging to a point ζ_0 of $\partial\Omega$ and some $\rho>0$, each component of the open set $\Omega\cap N_\rho(\zeta_0)$ contains only finitely many ζ_n. This assures us that for each $\delta>0$ there are infinitely many components of $\Omega\cap N_\rho(\zeta_0)$ intersecting $N_\delta(\zeta_0)$.

Now, let $\zeta=Z(t)$ $(0\leqq t\leqq 1)$ be a parametrization of $\partial\Omega$ as a closed curve. Starting with any point of $\partial\Omega$ in $N_\rho(\zeta_0)$ and tracing along $\partial\Omega$ in each direction to the first points of intersection with $C_\rho(\zeta_0)$, we obtain an open interval I_1 in $[0, 1]$ such that the arc $\Gamma_1=Z|I_1$ lies in $N_\rho(\zeta_0)$ with end points on $C_\rho(\zeta_0)$. Suppose next that $I_1, I_2, \cdots, I_{n-1}$ are disjoint open intervals in $[0, 1]$ for which the corresponding arcs $\Gamma_k=Z|I_k$ lie in $N_\rho(\zeta_0)$ with end points on $C_\rho(\zeta_0)$. The union of these arcs divides $N_\rho(\zeta_0)$ into disjoint regions, only finitely many of which have arcs of $C_\rho(\zeta_0)$ as part of their boundary. On the other hand, infinitely many components of $\Omega\cap N_\rho(\zeta_0)$ have this property and also intersect $N_{\rho/n}(\zeta_0)$. It is thus clear that we can find an open interval

I_n in $[0, 1]$ such that (i) $I_1, I_2, \cdots, I_n$ are disjoint, (ii) the arc $\Gamma_n = Z|I_n$ lies in $N_\rho(\zeta_0)$ with end points on $C_\rho(\zeta_0)$, and (iii) some point $Z(a_n)$ $(a_n \in I_n)$ of Γ_n satisfies $|Z(a_n) - \zeta_0| < \rho/n$.

An inductive procedure based on the above observations leads to the existence of a sequence of disjoint closed subintervals $[a_n, b_n]$ of $[0, 1]$ such that

$$Z(a_n) \to \zeta_0 \quad \text{and} \quad |Z(b_n) - \zeta_0| = \rho. \tag{2.1}$$

Extracting subsequences if necessary, we see that $\{a_n\}$ and $\{b_n\}$ can be chosen so as to converge to some point t_0 of $[0, 1]$. Since conditions (2.1) then contradict the continuity of Z at t_0, the lemma is established.

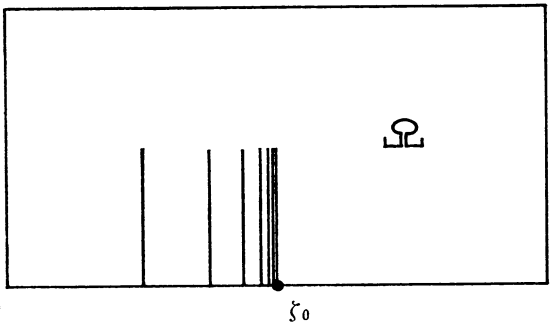

FIGURE 1

The converse is also true (see [2]), but we have no need for it here. An illustration of the content of the lemma is furnished by the example of Figure 1. For this region Ω the boundary is not parametrizable as a closed curve, and it is apparent that the boundary point ζ_0, for example, fails to be locally sequentially accessible. Note also that, even though all points of $\partial\Omega$ are accessible, no conformal mapping of ω onto Ω can be continuously extended to $\bar{\omega}$ (since such an extension would furnish a parametrization of $\partial\Omega$ as a closed curve).

The above lemma provides the key to our first extension theorem.

THEOREM 1. *Let Ω be a bounded simply connected plane region for which $\partial\Omega$ is parametrizable as a closed curve, and let χ be a homeomorphism of the open unit disc ω onto Ω. Then χ can be extended to a continuous mapping of $\bar{\omega}$ onto $\bar{\Omega}$ if and only if*

$$\liminf_{r \to 0} \sigma_r(z_0) = 0 \tag{2.2}$$

for each point z_0 of $\partial\omega$.

PROOF. The necessity of (2.2) is obvious, and we turn our attention to its sufficiency. Suppose that χ fails to have a limit at some point z_0 of $\partial\omega$. There will then exist sequences $\{z_n\}$ and $\{z_n'\}$ in ω converging to z_0 with the corresponding image sequences $\{\zeta_n\}$ and $\{\zeta_n'\}$ converging to distinct points ζ_0 and ζ_0', respectively, of $\partial\Omega$. Putting $\rho=|\zeta_0-\zeta_0'|/3$, we invoke the local sequential accessibility of ζ_0 to conclude that infinitely many ζ_n lie in some component, say $\Omega_\rho(\zeta_0)$, of $\Omega\cap N_\rho(\zeta_0)$. Without loss of generality all ζ_n will be presumed to lie in $\Omega_\rho(\zeta_0)$. Similarly, all ζ_n' will be presumed to lie in some component $\Omega_\rho(\zeta_0')$ of $\Omega\cap N_\rho(\zeta_0')$.

Taking account of (2.2), we choose r (>0) so small that (i) z_1 and z_1' fall outside $N_r(z_0)$ and (ii) $\sigma_r(z_0)<\rho$. We then fix n so large that z_n falls inside $N_r(z_0)$. Since ζ_1 can be joined to ζ_n by a curve in $\Omega_\rho(\zeta_0)$, and this is the image of some curve joining z_1 to z_n in ω, there is a point z of $\omega\cap C_r(z_0)$ whose image $\chi(z)$ lies in $N_\rho(\zeta_0)$. In the same way, some point z' of $\omega\cap C_r(z_0)$ has its image $\chi(z')$ in $N_\rho(\zeta_0')$. There results $\sigma_r(z_0)\geqq|\chi(z)-\chi(z')|>\rho$, contradicting condition (ii) on the choice of r, and the proof is complete.

The second extension theorem is a direct topological counterpart of the Osgood-Taylor-Carathéodory theorem.

THEOREM 2. *Let Ω be a plane region bounded by a Jordan curve, and let χ be a homeomorphism of the open unit disc ω onto Ω. If*

$$\liminf_{r\to 0}\sigma_r(z_0) = 0$$

for each point z_0 of $\partial\omega$, and if χ does not tend to a constant value on any subarc of $\partial\omega$, then χ can be extended to a homeomorphism of $\bar\omega$ onto $\bar\Omega$.

PROOF. Extending χ according to Theorem 1, so as to map $\bar\omega$ continuously onto $\bar\Omega$, we shall show that the resulting mapping is a homeomorphism. Suppose, in fact, that this is not the case, i.e. that two distinct points z_1 and z_2 of $\partial\omega$ are carried into the same point ζ_0 of $\partial\Omega$. Then the curve γ in $\bar\omega$ formed by the two radial segments joining the origin to z_1 and z_2 has as its image in $\bar\Omega$ a Jordan curve Γ intersecting $\partial\Omega$ in precisely the point ζ_0. One of the two regions into which γ divides ω, say ω_0, maps onto the region Ω_0 enclosed by Γ (the Jordan curve theorem is, of course, used here). Since the circular boundary arc γ_0 of ω_0 is a subset of both $\partial\omega_0$ and $\partial\omega$, its image is contained in $\Gamma\cap\partial\Omega=\{\zeta_0\}$, and we have a manifest contradiction to the assumed boundary behavior of χ.

3. **Applications to conformal and quasiconformal mappings.** Let χ be a C' homeomorphism of ω onto a bounded plane region Ω, and

let J be the Jacobian of χ. If, given z_0 on $\partial\omega$, there exists a constant K such that

$$\frac{1}{r^2}\left|\frac{d\chi(z_0+re^{i\theta})}{d\theta}\right|^2 \leqq KJ(z_0+re^{i\theta}) \tag{3.1}$$

for $0<r<\delta$, then the limiting oscillation condition (2.2) holds.

To prove this assertion, we take $\alpha(r)$ and $\beta(r)$, respectively, as the minimum and maximum values of θ (on an appropriate interval of length π) for which $z_0+re^{i\theta}$ lies in $\bar{\omega}$. Then, since the oscillation $\sigma_r(z_0)$ cannot exceed the length of the image in Ω of the arc $\omega\cap C_r(z_0)$, we have

$$\sigma_r(z_0) \leqq \int_{\alpha(r)}^{\beta(r)}\left|\frac{d\chi(z_0+re^{i\theta})}{d\theta}\right| d\theta.$$

An application of the Schwarz inequality in conjunction with (3.1) leads to

$$\int_0^\delta \frac{[\sigma_r(z_0)]^2}{r}\,dr \leqq \pi K\int_0^\delta\int_{\alpha(r)}^{\beta(r)} J(z_0+re^{i\theta})r\,d\theta\,dr \leqq \pi KA,$$

where A is the area of Ω, and condition (2.2) follows from the finiteness of the left-hand integral.

If χ is conformal, then

$$\frac{1}{r^2}\left|\frac{d\chi(z_0+re^{i\theta})}{d\theta}\right|^2 = |\chi'(z_0+re^{i\theta})|^2 = J(z_0+re^{i\theta}),$$

so that (3.1) holds with $K=1$. Theorem 1 thus yields

THEOREM 3. *Let Ω be a bounded simply connected plane region for which $\partial\Omega$ is parametrizable as a closed curve. If χ is a conformal mapping of the open unit disc ω onto Ω, then χ can be extended to a continuous mapping of $\bar{\omega}$ onto $\bar{\Omega}$.*

For conformal mappings, Theorem 2 reduces to the following equivalent form of the Osgood-Taylor-Carathéodory theorem.

THEOREM 4. *If χ is a conformal mapping of the open unit disc ω onto a region Ω bounded by a Jordan curve, then χ can be extended to a homeomorphism of $\bar{\omega}$ onto $\bar{\Omega}$.*

That the boundary property of χ required in Theorem 2 holds for conformal mappings is evident from a theorem of Painlevé, dating back to 1888, which states that a nonconstant analytic function cannot tend to a constant value along any boundary arc. In the present

setting, however, we have access to a particularly elementary direct argument. Suppose that the conformal mapping χ, extended according to Theorem 3, carries a boundary arc γ_0 of ω into a single point ζ_0 of $\partial\Omega$. Then the expression

$$u(z) = \log(d/|\chi(z) - \zeta_0|),$$

where d is the diameter of Ω, defines u as a positive harmonic function on ω tending to $+\infty$ on γ_0, and Fatou's lemma results in the contradiction

$$u(0) = \frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta})d\theta \to +\infty.$$

Let us conclude with some observations concerning the case of quasiconformal mappings. That the Osgood-Taylor-Carathéodory theorem remains valid for quasiconformal mappings has already been pointed out by Künzi [**5**, p. 68] in the C' case and by Lehto and Virtanen [**6**, pp. 44–46] in the general case. The two approaches rely on analogues of classical arguments for conformal mappings. As we proceed to show, however, quasiconformal extension theorems can actually be reduced at once to conformal extension theorems by appealing to a theorem of Mori.

The technique here is based on use of the following decomposition: if χ is any quasiconformal mapping of the disc ω onto a simply connected proper subregion Ω of the plane, then

$$\chi = T \circ \lambda, \tag{3.2}$$

where T is a conformal mapping of ω onto Ω and λ is a quasiconformal mapping of ω onto itself. This is obvious by starting with an arbitrary conformal mapping T of ω onto Ω and defining λ as the quasiconformal mapping $T^{-1} \circ \chi$. The theorem of Mori guarantees that any quasiconformal mapping λ of ω onto itself can be extended to a homeomorphism of $\bar{\omega}$ onto itself (see [**5**, p. 101] or [**6**, p. 69]). By extending T according to Theorems 3 and 4 it is now clear that both of these theorems remain in force when χ is allowed to be quasiconformal.

It should be noted also that condition (3.1) is known to hold for C' quasiconformal mappings (see e.g. [**5**, p. 23]), and this suffices to establish Theorem 3 for such mappings without recourse to the theorem of Mori. Similar considerations apply for general quasiconformal mappings, where (3.1) holds except on sets of measure zero, but rather delicate measure-theoretic results are needed (see [**6**, pp. 140, 173, 179]).

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