SHORTER NOTES

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A UNIQUENESS THEOREM FOR CERTAIN TWO-POINT BOUNDARY VALUE PROBLEMS

JAMES S. W. WONG

We are here concerned with establishing uniqueness of solutions for the following two-point boundary value problem:

$$x'' = f(t, x, x'), \qquad x(a) = A, \qquad x(b) = B, \tag{1}$$

with

$$f(t, x, u) - f(t, y, v) > g(t, x - y, u - v) \qquad \text{if } x > y, \tag{2}$$

where $g(t, z, p)$ satisfies the following: (a) the initial value problem:

$$z'' = g(t, z, z'), \qquad z(c) = 0, \quad z'(c) = C, \tag{3}$$

where $c \geqq a$ and C arbitrary, has a solution defined for all $t \geqq c$, (b) there exists a number $h > 0$ such that no nontrivial solution $z(t)$ of (3) may satisfy $z(c) = z(d) = 0$, with $d - c < h$, and (c) for any p,

$$g(t, z_1, p) \geqq g(t, z_2, p) \qquad \text{if } z_1 \geqq z_2. \tag{4}$$

Our main result is the following:

THEOREM. *Under the above assumptions, if $b - a \leqq h$, then* (1) *has at most one solution.*

PROOF. Suppose that $x_1(t)$ and $x_2(t)$ are two distinct solutions of (1), and write $\phi(t) = x_1(t) - x_2(t)$. Without loss of generality, we may assume that there exist numbers c, d such that $a \leqq c < d \leqq b$, $\phi(c) = \phi(d) = 0$ and $\phi(t) > 0$ for $t \in (c, d)$. Consider the solution of (3) with initial conditions $z(c) = 0$, $z'(c) = \phi'(c)$. Let $\psi(t) = \phi(t) - z(t)$. Clearly, $\psi(c) = \psi'(c) = 0$. From (1), (2), and (3), observe that

$$\begin{aligned}
\psi''(c) &= \phi''(c) - z''(c) \\
&= x_1''(c) - x_2''(c) - z''(c) \\
&= f(c, x_1(c), x_1'(c)) - f(c, x_2(c), x_2'(c)) - z''(c) \\
&> g(c, 0, x_1'(c) - x_2'(c)) - g(x, 0, z'(c)) \\
&= g(c, 0, \phi'(c)) - g(c, 0, z'(c)) = 0,
\end{aligned}$$

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hence one concludes that there exists $t_0 \in (c, d)$ such that $\psi(c) = \psi(t_0) = 0$ and $\psi(t) > 0$ for $t \in (c, t_0)$. Hence there must exist a number $t_1 \in (c, t_0)$ such that $\psi'(t_1) = 0$, and $\psi''(t_1) \leqq 0$. On the other hand, we note

$$\phi''(t_1) - z''(t_1) > g(t_1, \phi(t_1), \phi'(t_1)) - g(t_1, z(t_1), z'(t_1)) \geqq 0,$$

which is the desired contradiction.

In the special case when $f(t, x, x') = h(t) - k(x)$ and $g(t, y, y') = -y$ for some continuous functions $h(t)$ and $k(x)$, the above theorem simplifies the hypothesis and the proof of a recent result of Maric and Tomic [**1**] on the Duffing problem. Our result is somewhat related to the classical problem of estimating the interval of uniqueness of (1). In contrast to the traditional approach, we impose a strict inequality (namely (2)) and obtain a larger interval of uniqueness (i.e., $b - a \leqq h$). On the other hand, the traditional approach when $g(t, y, p)$ is linear in y and p, yields $b - a < h$ which does not cover the result of [**1**]. The fact that the one-sided inequality (2) plays an important role in the uniqueness theorem for (1) has also been explored by Bailey and Waltman [**2**], where other references may be found.

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ADDED IN PROOF. Cf. also, D. Willett, SIAM Review **9** (1967), 726–728.

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1. V. Maric and M. Tomic, *A contribution to Duffing's boundary value problem*, J. Math. Anal. Appl. **5** (1962), 147–149.

2. P. Bailey and P. Waltman, *Existence and uniqueness of solutions of the first boundary value problem for non-linear second order differential equations*, Arch. Rational Mech. Anal. **21** (1966), 310–320.

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