SHORTER NOTES

The purpose of this department is to publish very short papers of an unusually elegant and polished character, for which there is no other outlet.

A UNIQUENESS THEOREM FOR CERTAIN TWO-POINT BOUNDARY VALUE PROBLEMS

JAMES S. W. WONG

We are here concerned with establishing uniqueness of solutions for the following two-point boundary value problem:

\[(1) \quad x'' = f(t, x, x'), \quad x(a) = A, \quad x(b) = B,\]

with

\[(2) \quad f(t, x, u) - f(t, y, v) > g(t, x - y, u - v) \quad \text{if} \ x > y,\]

where \(g(t, z, p)\) satisfies the following: (a) the initial value problem:

\[(3) \quad z'' = g(t, z, z'), \quad z(c) = 0, \quad z'(c) = C,\]

where \(c \geq a\) and \(C\) arbitrary, has a solution defined for all \(t \geq c\), (b) there exists a number \(h > 0\) such that no nontrivial solution \(z(t)\) of (3) may satisfy \(z(c) = z(d) = 0\), with \(d - c < h\), and (c) for any \(p\),

\[(4) \quad g(t, z_1, p) \geq g(t, z_2, p) \quad \text{if} \ z_1 \geq z_2.\]

Our main result is the following:

THEOREM. Under the above assumptions, if \(b - a \leq h\), then (1) has at most one solution.

Proof. Suppose that \(x_1(t)\) and \(x_2(t)\) are two distinct solutions of (1), and write \(\phi(t) = x_1(t) - x_2(t)\). Without loss of generality, we may assume that there exist numbers \(c, d\) such that \(a \leq c < d \leq b\), \(\phi(c) = \phi(d) = 0\) and \(\phi(t) > 0\) for \(t \in (c, d)\). Consider the solution of (3) with initial conditions \(z(c) = 0, z'(c) = \phi'(c)\). Let \(\psi(t) = \phi(t) - z(t)\). Clearly, \(\psi(c) = \psi'(c) = 0\). From (1), (2), and (3), observe that

\[\psi''(c) = \phi''(c) - z''(c)\]
[\[= x_1''(c) - x_2''(c) - z''(c)\]
[\[= f(c, x_1(c), x_1'(c)) - f(c, x_2(c), x_2'(c)) - z''(c)\]
[\[> g(c, 0, x_1'(c) - x_2'(c)) - g(x, 0, z'(c))\]
[\[= g(c, 0, \phi'(c)) - g(c, 0, z'(c)) = 0,\]

Received by the editors November 28, 1966.
hence one concludes that there exists $t_0 \in (c, d)$ such that $\psi(c) = \psi(t_0) = 0$ and $\psi(t) > 0$ for $t \in (c, t_0)$. Hence there must exist a number $t_1 \in (c, t_0)$ such that $\psi'(t_1) = 0$, and $\psi''(t_1) \leq 0$. On the other hand, we note

$$\phi''(t_1) - z''(t_1) > g(t_1, \phi(t_1), \phi'(t_1)) - g(t_1, z(t_1), z'(t_1)) \geq 0,$$

which is the desired contradiction.

In the special case when $f(t, x, x') = h(t) - k(x)$ and $g(t, y, y') = -y$ for some continuous functions $h(t)$ and $k(x)$, the above theorem simplifies the hypothesis and the proof of a recent result of Maric and Tomic [1] on the Duffing problem. Our result is somewhat related to the classical problem of estimating the interval of uniqueness of (1). In contrast to the traditional approach, we impose a strict inequality (namely (2)) and obtain a larger interval of uniqueness (i.e., $b - a \leq h$). On the other hand, the traditional approach when $g(t, y, p)$ is linear in $y$ and $p$, yields $b - a < h$ which does not cover the result of [1]. The fact that the one-sided inequality (2) plays an important role in the uniqueness theorem for (1) has also been explored by Bailey and Waltman [2], where other references may be found.

**Acknowledgement.** Thanks are due to J. W. Macki for several stimulating discussions and D. Willett for reviving my interest in this topic.

**Added in Proof.** Cf. also, D. Willett, SIAM Review 9 (1967), 726–728.

**References**


**University of Alberta**