GENERALIZED EIGENFUNCTIONS OF THE LAPLACE OPERATOR AND WEIGHTED AVERAGE PROPERTY

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I. Introduction. In the previous two papers, [1] and [2], we were interested in characterizing the class \( S(w, R) \) of real-valued functions \( u \), defined in a given region (open, connected set) \( R \) of the \( n \)-dimensional Euclidean space \( E_n \) which satisfy the Weighted Average Property (W.A.P.):

\[
(1) \quad u(P) = \frac{\int_{B(P, r)} u \cdot w \, dp}{\int_{B(P, r)} w \, dp}, \quad P \in R,
\]

where \( B(P, r) \) denotes any ball with the point \( P = (x_1, x_2, \ldots, x_n) \) for its center and radius \( r \) whose closure lies in \( R \); \( dp \) stands for the usual Lebesgue measure of \( B \) and \( w \) is a weight function (W.F.) defined in \( R \) (i.e., \( w \) is nonnegative and locally summable in \( R \)).

It was proved in paper [1] that \( S(w, R) \) is always a subspace of the solution space of the second-order linear elliptic homogeneous differential equation:

\[
(2) \quad w \Delta u + 2 \sum_{i=1}^{n} u_{x_i} w_{x_i} = 0
\]

in \( R \), where \( u \) is the Laplacian of \( u \), provided \( w \in \mathcal{C}^1(R) \). Furthermore, \( S(w, R) \) is the solution space of the equation (2) if the W.F. \( w \) is an eigenfunction of the Laplace operator. That is, if \( w \) is a solution of an equation of the form:

\[
(3) \quad \Delta w + \lambda w = 0
\]

in \( R \), where \( \lambda \) is some real constant." In the latter case \( S(w, R) \) is infinite dimensional."

The present paper is an extension of these results. In this paper we want to prove that \( S(w, R) \) is always a subspace of the (common) solution space of a system of equations of the form:

\[
(4) \quad \Delta^k w \cdot \Delta u + 2 \sum_{i=1}^{n} u_{x_i}(\Delta^k w)_{x_i} = 0, \quad k = 0, 1, 2, 3, \ldots
\]

where \( \Delta^k w \) is the \( k \)th iteration of the Laplacian of \( w \) (\( \Delta^0 w \) is interpreted as \( w \)), provided the weight function \( w \) is sufficiently differentiable."
Furthermore, "$S(w, R)$ is precisely the solution space of the system of equations (4), provided the weight function $w$ is a generalized eigenfunction of the Laplace operator or more generally a weight function satisfying an equation of the form:

$$\Delta^m w = a_0 w + a_1 \Delta w + \cdots + a_{m-1}\Delta^{m-1}w,$$

in the region $R$, where $m$ is a positive integer and the $a_i$'s are real constants."

In paper [2] it was proved that "in $E_2$, $S(w, R)$ is infinite dimensional if and only if $w$ is an eigenfunction of the Laplace operator; otherwise, $S(w, R)$ is finite dimensional and $1 \leq \dim S(w, R) \leq 2$." It was also claimed there that in $E_n$, $n > 2$, "$S(w, R)$ is infinite dimensional if and only if $w$ is an eigenfunction of the Laplace operator; otherwise, $S(w, R)$ is finite dimensional and $1 \leq \dim S(w, R) \leq 2n-1." In a footnote of [2] it was mentioned that the above statement is, possibly, not true in $E_n$, $n > 2$. The main result of this paper clarifies this point completely. Indeed, it will be shown in §III of this paper that in $E_n$, $n > 2$, $S(w, R)$ could be infinite dimensional even if the weight function $w$ is not an eigenfunction of the Laplace operator.

II. THEOREM 1. Let $R$ be a region in $E_n$ and $w$ be a W. F. defined in $R$.

(i) If $w \in C^{m+1}(R)$, $m$ being a nonnegative integer, then $S(w, R)$ is a subspace of the solution space of the system of equations:

$$\Delta^k w \cdot \Delta u + 2 \sum_{i=1}^n u_{x_i}(\Delta^k w)_{x_i} = 0, \quad \text{in } R, \quad k = 0, 1, 2, \ldots, m.$$  

(ii) If $\Delta^k w$ is defined and is in $C^1(R)$ for each positive integer $k$, then $S(w, R)$ is a subspace of the solution space of the system of equations:

$$\Delta^k w \cdot \Delta u + 2 \sum_{i=1}^n u_{x_i}(\Delta^k w)_{x_i} = 0 \quad \text{in } R, \quad k = 0, 1, 2, 3, \ldots, \infty$$

Proof. Part (i). Let $u \in S(w, R)$. Then by Theorem 2 and Theorem 4 of [1], $u \in C^{m+2}(R)$ and

$$w\Delta w + 2 \sum_{i=1}^n u_{x_i}w_{x_i} = 0 \quad \text{in } R.$$  

Also by Theorem 3 of [1], we get the circumferential mean-value property

$$\int_{S(P, r)} u w d\sigma = u(P) \int_{S(P, r)} w d\sigma,$$
for each $S(P, r)$—the boundary of $B(P, r)$—which, together with its interior $B(P, r)$, lies in $R$. As proved in Theorem 4 of [1], we get immediately from (8), by differentiating with respect to $r$, the mean-value relation

$$u(P) \int_{S(P, r)} \frac{\partial w}{\partial n} \, d\sigma = \int_{S(P, r)} \frac{\partial uw}{\partial n} \, d\sigma$$

for each $S(P, r)$ which, together with its interior $B(P, r)$, lies in $R$, where $\partial / \partial n$ refers to the derivative in the direction of the outward drawn normal to the surface $S(P, r)$. Now, using Green's formula and relations (7) and (9), we get

$$\int_{B(P, r)} \left( w\Delta u + 2 \sum_{i=1}^n u_{x_i} w_{x_i} + u\Delta w \right) \, d\rho = \int_{B(P, r)} \Delta (uw) \, d\rho$$

$$\quad = \int_{S(P, r)} \frac{\partial uw}{\partial n} \, d\sigma = u(P) \int_{S(P, r)} \frac{\partial w}{\partial n} \, d\sigma$$

or

$$\int_{B(P, r)} u\Delta wd\rho = u(P) \int_{B(P, r)} \Delta wd\rho,$$

for each $B(P, r)$ whose closure lies in $R$. It is now clear that we can apply the entire reasoning of Theorem 4 of [1] over again to the averaging property (10) and get

$$\Delta w \cdot \Delta u + 2 \sum_{i=1}^n u_{x_i}(\Delta w)_{x_i} = 0 \quad \text{in} \; R.$$

Since $\Delta^k w$ is defined and is in class $C^1(R)$ for $k=0, 1, 2, \cdots, m$, repeating the argument a finite number of times, we see that $u$ must satisfy the system of equations:

$$\Delta^k w \cdot \Delta u + 2 \sum_{i=1}^n u_{x_i}(\Delta^k w)_{x_i} = 0, \quad k = 0, 1, 2, \cdots, m \quad \text{in} \; R.$$

This proves Part (i).

Part (ii). Let $u \in S(w, R)$. Since $\Delta^k w$ is defined and is in class $C^1(R)$ for all $k=0, 1, 2, 3, \cdots$, the above reasoning together with mathematical induction implies that $u$ must be a common solution of the system of equations:

$$\Delta^k w \cdot \Delta u + 2 \sum_{i=1}^n u_{x_i}(\Delta^k w)_{x_i} = 0$$
in \( R \), \( k = 0, 1, 2, 3, \ldots, \infty \). This proves Part (ii).

**Theorem 2.** Defined in a region \( R \) of \( E_n \), let \( w \) be a W. F. belonging to class \( C^{2m}(R) \) and be a solution of the differential equation

\[
\Delta^m w = a_0 w + a_1 \Delta w + \cdots + a_{m-1} \Delta^{m-1} w
\]

in \( R \), where \( m \) is a positive integer and the \( a_i \)'s are real constants. A real-valued function \( u \) is in \( S(w, R) \) if and only if \( u \) is in \( C^2(R) \) and is a common solution of the system of equations:

\[
\Delta^k u \cdot \Delta u + 2 \sum_{i=1}^{n} u_{x_i}(\Delta^k w)_{x_i} = 0, \quad k = 0, 1, 2, \ldots, m - 1, \text{ in } R.
\]

**Proof.** Let \( u \in S(w, R) \). By Theorem 2 of [1], \( u \in C^{2m+1}(R) \). Since \( w \in C^{2m}(R) \), the right-hand side of (12) is in \( C^2(R) \). Hence, \( \Delta^{m+1} w \) is defined in \( R \) and is a linear combination of \( w, \Delta w, \Delta^2 w, \cdots, \Delta^{m-1} w \) in \( R \). Applying mathematical induction, it is easy to see that \( \Delta^{m+p} w \) is defined in \( R \) and \( \Delta^{m+p+1} w \) is a linear combination of \( w, \Delta w, \Delta^2 w, \cdots, \Delta^{m-1} w \) in \( R \) for each positive integer \( p \). This means that \( \Delta^k w \) is defined in \( R \) and is a linear combination of \( w, \Delta w, \Delta^2 w, \cdots, \Delta^{m-1} w \) in \( R \) for each positive integer \( k \). Hence, by Part (ii) of Theorem 1, it follows that \( u \) is a common solution of the system of equations

\[
\Delta^k w \cdot \Delta u + 2 \sum_{i=1}^{n} u_{x_i}(\Delta^k w)_{x_i} = 0
\]

in \( R \), \( k = 0, 1, 2, 3, \ldots, \infty \). Hence, \( u \) is a common solution of system (13) of equations.

Conversely, suppose that \( u \in C^2(R) \) and is a common solution of the system of equations (13) in \( R \). Since \( \Delta^k w \) is defined in \( R \) and is a linear combination of \( w, \Delta w, \Delta^2 w, \cdots, \Delta^{m-1} w \) for each positive integer \( k \), it follows that every member of the system (14) of equations can be expressed as a linear combination of the \( m \) equations of the system (13). That is, \( u \) is also a common solution of the system (14) of equations in \( R \). Since \( \Delta(\Delta w) \) is defined in \( R \) and \( u \) is a solution of

\[
w \Delta u + 2 \sum_{i=1}^{n} u_{x_i} w_{x_i} = 0 \text{ in } R,
\]

we have

\[
\Delta(\Delta w) = w \Delta u + 2 \sum_{i=1}^{n} u_{x_i} w_{x_i} + u \Delta w = u \Delta w \text{ in } R.
\]
Now, assuming that $\Delta^k(\omega w)$ is defined and $\Delta^k(\omega w) = u\Delta^k w$ in $R$ for some positive integer $k$ and using the fact that $u$ is a solution of the system (14) of equations in $R$, we see that $\Delta^{k+1}(\omega w)$ is defined in $R$ and

$$\Delta^{k+1}(\omega w) = \Delta^k w \cdot \Delta u + 2 \sum_{i=1}^{n} u_{x_i}(\Delta^k w)_{x_i} + u \Delta^{k+1} w = u \Delta^{k+1} w \quad \text{in } R.$$ 

Hence, by mathematical induction $\Delta^k(\omega w)$ is defined in $R$ and

$$\Delta^k(\omega w) = u \cdot \Delta^k w$$

in $R$ for each positive integer $k$. Putting $k = m$ and using (12), we see that

$$\Delta^m(\omega w) = u \Delta^m w = a_0(\omega w) + a_1 \Delta(\omega w) + \cdots + a_{m-1} \Delta^{m-1}(\omega w).$$

Therefore, $\omega w$ is also a solution of equation (12) in $R$. Let $B(P, r)$ be any ball which, together with its boundary $S(P, r)$, lies in $R$. As proved in [3, pp. 286–289],

$$\frac{1}{\Omega_r} \int_{S(P, r)} \omega d\sigma = \Gamma(n/2) \sum_{r=0}^{m+k-1} (r/2)^{2r} \frac{\Delta^r w(P)}{r! \Gamma(v + n/2)}$$

$$+ \int_{B(P, r)} v_{m+k}(\beta) \Delta^{m+k} w d\rho$$

for all positive integers $k$, where $\Omega_r$ is the surface area of $S(P, r)$ and the sequence of functions $v_m(\beta)$ are given by the recursion system

$$v_{r+1}(\beta) = \frac{1}{(n-2)\beta^{n-2}} \int_{\beta}^{r} \alpha v_r(\alpha) (\alpha^{n-2} - \beta^{n-2}) d\alpha,$$

$$v_0(\beta) = (1/(n-2) \Omega_1) (1/\beta^{n-2} - 1/r^{n-2}),$$

$\Omega_1$ being the surface area of the unit sphere in $n$-dimensional space and $\beta$ being the distance of a point in $B(P, r)$ from the center $P$. [In two dimensions the recursion formula (17) is given by

$$v_{r+1}(\beta) = \int_{\beta}^{r} \alpha v_r(\alpha) \log \alpha/\beta d\alpha,$$

$$v_0(\beta) = (1/2\pi) \log r/\beta.$$

Since $\Delta^m w = a_0 w + a_1 \Delta w + \cdots + a_{m-1} \Delta^{m-1} w$, we have

$$\Delta^{m+k} w = a_0^{(k)} w + a_1^{(k)} \Delta w + \cdots + a_{m-1}^{(k)} \Delta^{m-1} w, \quad k = 1, 2, 3, \ldots, \infty,$$

where the sequences of constants $\{a_i^{(k)}\}, \ i = 0, 1, 2, \cdots, m-1$, are defined recursively by
\begin{align*}
a^{(k)}_0 &= a^{(k-1)}_{m-1} a_0, \\
a^{(k)}_1 &= a^{(k-1)}_0 + a^{(k-1)}_{m-1} a_1, \ldots, \\
a^{(k)}_{m-1} &= a^{(k-1)}_{m-2} + a^{(k-1)}_{m-1} a_{m-1},
\end{align*}

\(a^{(0)}_i\) being interpreted as \(a_i, i = 0, 1, \ldots, m-1.\)

Let \(c/2\) be a positive number greater than each of the numbers 1, \(|a_0|, |a_1|, \ldots, |a_{m-1}|\). Then it is easy to see that
\[
|a^{(k)}_i| < c^{k+1}
\]
for all \(i = 0, 1, 2, \ldots, m-1,\) and for all nonnegative integers \(k.\) Also, the \(m\) functions \(w, \Delta w, \Delta^2 w, \ldots, \Delta^{m-1} w\) are each bounded by a positive constant \(\lambda\) on the closure of \(B(P, r).\) Now, it has already been shown in [3, p. 289] that the remainder term \(\int_{B(P, r)} v_{m+k} \Delta^{m+k} w d\rho\) in (16) tends to zero as \(k\) tends to \(\infty,\) provided \(w \in C^2(R)\) and is a solution of \(\Delta w - cw = 0\) in \(R,\) where \(c\) is a positive constant. Since \(g = \exp(c/n)^{1/2} \sum_{i=1}^n x_i\) is in \(C^2(R)\) and satisfies \(\Delta w - cw = 0\) in \(R,\) where \(c\) is given by (19), we have
\[
\lim_{k \to \infty} \int_{B(P, r)} v_{m+k} \Delta^{m+k} g d\rho = 0.
\]

Let \(g_0\) be the minimum of \(g\) on the closure of \(B(P, r).\) Then \(g_0 > 0\) and
\[
0 \leq c^{m+k} g_0 \int_{B(P, r)} v_{m+k} d\rho \leq \int_{B(P, r)} v_{m+k} c^{m+k} g d\rho
\]
\[
= \int_{B(P, r)} v_{m+k} \Delta^{m+k} g d\rho.
\]

From (20) and (21) we see that
\[
\lim_{k \to \infty} c^{m+k} \int_{B(P, r)} v_{m+k} d\rho = 0.
\]

Now, returning to the relation (16), where \(w\) satisfies the equation (12), we see that
\[
0 \leq \left| \int_{B(P, r)} v_{m+k} \Delta^{m+k} w d\rho \right| \leq \int_{B(P, r)} v_{m+k} \left| \Delta^{m+k} w \right| d\rho
\]
\[
\leq (m\lambda/c^{m-1}) c^{m+k} \int_{B(P, r)} v_{m+k} d\rho.
\]

From (22) and (23) it now follows that the remainder term \(\int_{B(P, r)} v_{m+k} \Delta^{m+k} w d\rho\) in (16) tends to zero as \(k\) tends to \(\infty.\) Hence, we have

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Similarly,

\[
\frac{1}{\Omega_r} \int_{S(P,r)} w d\sigma = \Gamma(n/2) \sum_{\nu=0}^{\infty} \frac{\Delta^\nu w(P)}{\nu!\Gamma(\nu + n/2)} \cdot
\]

From (24) and (25) we get

\[
\int_{S(P,r)} u w d\sigma = u(P) \int_{S(P,r)} w d\sigma,
\]

for each \( S(P, r) \) which together with its interior \( B(P, r) \) lies in \( R \). This means by Theorem 3 of [1] that \( u \in S(w, R) \). This completes the proof.

III. As mentioned in the Introduction, we will now give an example to show that in \( E_n, n > 2, S(w, R) \) can be infinite dimensional even if the W. F. \( w \) is not an eigenfunction of the Laplace operator.

In \( E_3 \) let us consider the following W. F.:

\[
w(x_1, x_2, x_3) = e^{x_1} + e^{x_2} \quad \text{for all} \quad (x_1, x_2, x_3) \in E_3.
\]

Clearly, \( w \) is not an eigenfunction of the Laplace operator. We have

\[
\Delta^2 w = 5\Delta w - 4w, \quad w_{x_1} = w_{x_2} = (\Delta w)_{x_1} = (\Delta w)_{x_2} = 0
\]

in \( E_3 \). Hence, by Theorem 2, \( S(w, E_3) \) is the solution space of the system of elliptic equations:

\[
w \Delta u + 2 \sum_{i=1}^{3} u_{x_i} w_{x_i} = 0,
\]

\[
\Delta w \cdot \Delta u + 2 \sum_{i=1}^{3} u_{x_i} (\Delta w)_{x_i} = 0
\]

in \( E_3 \).

Now, for each positive integer \( k \), let \( P_k(x_1, x_2) \) be a harmonic polynomial of degree \( k \) in \( E_2 \). Then each of the functions

\[
f_k(x_1, x_2, x_3) = P_k(x_1, x_2), \quad k = 1, 2, 3, \ldots, \infty,
\]

is defined in \( E_3 \) and is harmonic in \( E_3 \). Hence,

\[
\Delta f_k = 0 = (f_k)_{x_3}
\]

in \( E_2 \), for all positive integers \( k \). It is clear that each of the functions \( f_k \) is a solution of system (24) of equations. Therefore, \( f_k \in S(w, E_3) \).
for each positive integer $k$. Since the infinite set of functions 
\{\{f_1, f_2, \cdots, f_k, \cdots\}\} is linearly independent over $E_3$, it follows that 
$S(w, E_3)$ is infinite dimensional.

It is also clear that similar types of examples can be constructed in 
four- and higher-dimensional Euclidean spaces.

**Bibliography**


2. ———, *Functions satisfying a weighted average property. II*, Trans. Amer. 


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