FUNCTIONS WHOSE FOURIER SERIES CONVERGE FOR EVERY CHANGE OF VARIABLE

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1. A theorem of Pál and Bohr, [1], [2], asserts that for every continuous f of period 2π there is a homeomorphism g of $[-\pi, \pi]$ with itself such that the Fourier series of $f \circ g$ converges uniformly. Salem [3] has given a powerful test for the uniform convergence of a Fourier series. On the other hand, there is no criterion which gives necessary and sufficient conditions for the Fourier series of a continuous function to converge everywhere.

In this note we show that the method of Salem may be used to determine a necessary and sufficient condition that a continuous function f be such that the Fourier series of $f \circ g$ should converge everywhere for every homeomorphism g of $[-\pi, \pi]$ with itself.

It is clear that this condition must be weaker than bounded variation since the continuous functions of bounded Φ-variation with $\Phi = t^p$, $p > 1$, have uniformly convergent Fourier series, and this class of functions is preserved by composition with homeomorphisms [3], [4].

2. We define right and left systems of intervals (at a point). Let $\{k_n\}$ be a sequence of positive integers such that $\lim_n k_n = \infty$ and $\lim_n k_n/n = 0$.

For each n, let I_{nm}, $m = 1, \cdots, k_n$, be disjoint closed intervals such that for each n, $I_{n,m-1}$ is to the left of I_{nm}. Let there be a real x such that for every $\epsilon > 0$ there is an N such that $I_{nm} \subset (x, x+\epsilon)$ whenever $n > N$. Then the collection $\mathcal{I} = \{I_{nm}: n = 1, 2, \cdots; m = 1, 2, \cdots, k_n\}$ is called a right system of intervals (at x). A left system is defined similarly.

3. Let f be a continuous function of period 2π and x be a point. It is easily seen that there is a sequence $\delta_n \searrow 0$ such that $\lim_n n\delta_n = \infty$ and

$$\lim_n \int_{\delta_n}^{\pi} f(x \pm t) D_n(t) dt = 0.$$

We need only note that if $\Delta_k \searrow 0$ then there exists N_k such that

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$$\left| \int_{\Delta_k}^{\pi} f(x \pm t) D_n(t) dt \right| < \Delta_k$$

whenever $n > N_k$. If m_k is an increasing sequence of integers with $m_k > N_k$, we may set $\delta_n = \Delta_k$ for $n \in (m_k, m_{k+1}]$. We may choose m_k increasing so rapidly that

$$\lim_n n\delta_n \geqq \lim_k \Delta_k m_k = \infty .$$

It is trivial that $\lim_n n\delta_n = \infty$ implies $\lim_n \int_{\delta_n}^{\pi} D_n(t)dt = 0$.

We have, consequently, that for $\phi(t) = f(x+t) - f(x)$ and $\psi(t) = f(x-t) - f(x)$, there is a sequence $\delta_n \searrow 0$ with $\lim_n n\delta_n = \infty$ such that

$$\lim_n \int_{\delta_n}^{\pi} \phi(t) \frac{\sin nt}{t} dt = 0,$$

$$\lim_n \int_{\delta_n}^{\pi} \psi(t) \frac{\sin nt}{t} dt = 0.$$

4. We introduce the symbol $\sum(f, k, n, \theta)$ to denote

$$\sum_{i=1}^{k} \frac{1}{i} [f((2i)\pi/n + \theta) - f((2i - 1)\pi/n + \theta)],$$

By combining an argument similar to that of Salem with the information available in the previous section, we obtain the following result:

For every f and x, there is a sequence $\{\theta_n\}$ with $0 < \theta_n < \pi/n$ and a sequence of integers $\{k_n\}$ with $\lim_n k_n = \infty$ and $\lim k_n/n = 0$ such that

$$\int_0^{\pi} \phi(t) \frac{\sin nt}{t} dt$$

and

$$(1/\pi) \sum (f, k_n, n, x + \theta_n)$$

are equiconvergent.

A similar result holds for the left side of x.

A suitable choice of k_n would be, for example, $[n\delta_n/2\pi - 1/2]$.

5. This result yields a criterion for everywhere convergence of Fourier series which is invariant under changes in variable.

For every right system, $\mathcal{I}$, consider the sequence

$$\alpha_n(\mathcal{I}) = \sum_{i=1}^{k_n} \frac{1}{i} f(I_{ni})$$

where, for any interval $I=[a, b]$, $f(I)=f(b)-f(a)$. $\alpha_n(\mathcal{I})$ is defined similarly for left systems.

Suppose now that f is such that $\alpha_n(\mathcal{I})$ converges to zero for every right and left system. Then, for every x, we may shrink the intervals $[x+\theta_n+(2i-1)\pi/n, x+\theta_n+2i\pi/n]$, $i=1, \cdots, k_n$, to obtain disjoint closed intervals I_{ni}, $i=1, \cdots, k_n$, so that $\alpha_n(\mathcal{I})$ is equiconvergent with $\sum(f, k_n, n, x+\theta_n)$. This yields the following result:

If f is such that $\alpha_n(\mathcal{I})$ converges to zero for every right and left system $\mathcal{I}$ then, for every homeomorphism g, the Fourier series of $f\circ g$ converges everywhere.

6. We now show that if f is such that there is a right (or left) system at a point x for which $\lim_n \alpha_n(\mathcal{I})\neq 0$, then there is a homeomorphism g such that the Fourier series of $f\circ g$ diverges at x. We assume, as we may, that $\mathcal{I}$ is a right system, $x=0$, $f(0)=0$ and for some finite $\alpha>0$, $\lim_n \sup \alpha_n(\mathcal{I})\geqq\alpha$.

We first describe briefly the essential idea behind the construction. Let $I_{ni}=[\tau_{ni}, \tau'_{ni}]$ for $I_{ni}\in\mathcal{I}$. We choose an increasing sequence of integers $\{m_n\}$ such that $(2k_{m_{n+1}}+1)/m_{n+1}<1/m_n$, $\tau'_{m_{n+1}k_{m_{n+1}}}<\tau_{m_n1}$, and $\alpha_{m_n}>\alpha(1-1/2^n)$. If g is constant on each interval $(i\pi/m_n, (i+1)\pi/m_n)$, $i=1, \cdots, 2k_{m_n}$, assuming the values τ_{m_n1}, τ'_{m_n1}, τ_{m_n2}, $\tau'_{m_n2}, \cdots$ successively, then, for $\{m_n\}$ increasing sufficiently rapidly, we can define g as a piecewise constant, nondecreasing function on $[0, \pi]$ so that

$$\int_0^{(2k_{m_n}+1)\pi/m_n} (f\circ g)(t)\,\frac{\sin m_n t}{t}\,dt$$

and α_{m_n}/π are equiconvergent and

$$\lim_n \int_{(2k_{m_n}+1)\pi/m_n}^{\pi} (f\circ g)(t)\,\frac{\sin m_n t}{t}\,dt = 0.$$

The g thus defined can be altered to give a piecewise linear, continuous, increasing function which has small slope on most of each of the former intervals of constancy and rises abruptly on a small portion of each of these intervals. This can be done so as to preserve the properties of g noted above.

It is somewhat simpler, however, to construct the homeomorphism g directly. This is accomplished by the use of the following

LEMMA. *Let $\{k_n\}$ be a sequence of integers with the properties $\lim_n k_n=\infty$, $\lim_n k_n/n=0$. There is a sequence $\{\epsilon_n\}$, $0<\epsilon_n<\pi/n$, such that for every function h continuous in a neighborhood of zero with*

$h(0)=0$ there is a sequence $\{\theta_n\}$, $0<\theta_n<\pi/n-\epsilon_n$, such that

$$\int_0^{(2k_n+1)\pi/n} h(t)\,\frac{\sin nt}{t}\,dt$$

and $(1/\pi)\sum(h, k_n, n, \theta_n)$ are equiconvergent.

PROOF. We write

$$\int_0^{(2k_n+1)\pi/n} h(t)\,\frac{\sin nt}{t}\,dt = \int_0^{\pi/n} \cdots + \sum_{i=1}^{2k_n} \int_{i\pi/n}^{(i+1)\pi/n} \cdots = P + Q.$$

We have

$$|P| \leqq C \sup_{t\in(0,\pi/n)} |h(t)| = o(1)$$

as $n\to\infty$ and, for small $\eta_n\in(0, \pi)$,

$$\begin{aligned} Q &= \int_0^{\pi} \sin t \sum_{i=1}^{2k_n} (-1)^i \frac{h((t+i\pi)/n)}{t+i\pi}\,dt \\ &= \int_0^{\pi-\eta_n} \cdots + \int_{\pi-\eta_n}^{\pi} \cdots \\ &= (1+\cos\eta_n) \sum_{i=1}^{2k_n} (-1)^i \frac{h(\theta_n + i\pi/n)}{n\theta_n + i\pi} + R \end{aligned}$$

where $\theta_n\in(0, (\pi-\eta_n)/n)$. Now

$$|R| \leqq \eta_n^2 \cdot \frac{1}{\pi} \sum_{i=1}^{2k_n} \frac{1}{i} \cdot \sup_{t\in(0,(2k_n+1)\pi/n)} |h(t)| = o(1)$$

as $n\to\infty$ if $\eta_n = O(1/\log k_n)^{1/2})$ since $k_n/n\to 0$ as $n\to\infty$.

We note now that

$$\begin{aligned} &\frac{h(\theta_n + 2i\pi/n)}{n\theta_n + 2i\pi} - \frac{h(\theta_n + (2i-1)\pi/n)}{n\theta_n + (2i-1)\pi} \\ &\qquad = (h(\theta_n + 2i\pi/n) - h(\theta_n + (2i-1)\pi/n))/(n\theta_n + 2i\pi) \\ &\qquad\quad - h(\theta_n + (2i-1)\pi/n)\pi/(n\theta_n + (2i-1)\pi)(n\theta_n + 2i\pi). \end{aligned}$$

If we drop the sum of the last terms, the error is less than

$$C \sup_{t\in(0,(2k_n+1)\pi/n)} |h(t)| = o(1)$$

as $n\to\infty$. Next we have

$$|1/(n\theta_n + 2i\pi) - 1/2i\pi| < 1/4\pi i^2$$

and so if we replace $n\theta_n+2i\pi+2i\pi$ by $2i\pi$, the error is again less than

$$C \sup_{t\in(0,(2k_n+1)\pi/n)} |h(t)| = o(1)$$

as $n\to\infty$. If we replace $(1+\cos\eta_n)$ by 2 we introduce an error of less than

$$(1-\cos\eta_n)\cdot\frac{1}{\pi}\left|\sum_{i=1}^{k_n}(h(\theta_n+2i\pi/n)-h(\theta_n+(2i-1)\pi/n))/2i\right|$$
$$< C\eta_n^2\log k_n \sup_{t\in(0,(2k_n+1)\pi/n)} |h(t)| = o(1)$$

as $n\to\infty$ if $\eta_n=O(1/(\log k_n)^{1/2})$.

Letting $\epsilon_n=\eta_n/n=O(1/n\ (\log k_n)^{1/2})$ the proof of the lemma is complete. We proceed with the construction of the homeomorphism g.

We choose an integer m_1 such that

$$(\mathrm{i}_1) \qquad \alpha_{m_1}(\mathcal{G}) > \alpha/2$$

and so large that for some $b_1\in(\tau'_{m_1k_{m_1}},\pi)$ and sufficiently close to π we have

$$(\mathrm{ii}_1) \qquad \left|\int_{(2k_{m_1}+1)\pi/m_1}^{\pi}(f\circ g)(t)\frac{\sin m_1t}{t}dt\right| < 1/2,$$

where g is the increasing linear function mapping $[(2k_{m_1}+1)\pi/m_1, \pi]$ onto $[b_1, \pi]$. To show that such a choice of m_1 and b_1 is possible, we note that the integral in (ii_1) is less than

$$|f(\pi)|\left|\int_{(2k_{m_1}+1)\pi/m_1}^{\pi}\frac{\sin m_1t}{t}dt\right| + \sup_{t\in(b_1,\pi)}|f(\pi)-f(t)|\,C\log m_1.$$

The first term can be made small by choosing m_1 large and, for any choice of m_1, the second term will be small if b_1 is sufficiently close to π.

We now set

$$g((2i-1)\pi/m_i) = \tau_{m_1i}, \quad g(2i\pi/m_1) = \tau'_{m_1i}, \qquad i=1,\cdots,k_{m_1},$$

and let g be linear on each interval $[i\pi/m_1, (i+1)\pi/m_1-\epsilon_{m_1}]$, $i=1,\cdots,2k_{m_1}$, and on each interval $[i\pi/m_1-\epsilon_{m_1}, i\pi/m_1]$, $i=1,\cdots,2k_{m_1}+1$, and be continuous and increasing on $[\pi/m_1,\pi]$. The slope of g on each interval $[i\pi/m_1, (i+1)\pi/m_1-\epsilon_{m_1}]$ is chosen so small that

$$(\mathrm{iii}_1) \qquad \left|\alpha_{m_1}(\mathcal{G}) - \sum(f\circ g, k_{m_1}, m_1, \theta)\right| < 1/2$$

for every $\theta\in(0, \pi/m_1-\epsilon_{m_1})$.

Suppose now that we have defined $m_1 < \cdots < m_r$ and a homeomorphism g from $[\pi/m_r, \pi]$ onto $[\tau_{m_r1}, \pi]$, $g(\pi)=\pi$, such that for $n=1, \cdots, r$ we have

$$(\mathrm{i}_n) \qquad \alpha_{m_n}(\mathcal{I}) > \alpha(1 - 1/2^n),$$

$$(\mathrm{ii}_n) \qquad \left| \int_{(2k_{m_n}+1)\pi/m_n}^{\pi} (f \circ g)(t) \frac{\sin m_n t}{t} \, dt \right| < 1/2^n,$$

and

$$(\mathrm{iii}_n) \quad \left| \alpha_{m_n}(\mathcal{I}) - \sum (f \circ g, k_{m_n}, m_n, \theta) \right| < 1/2^n, \quad \forall \theta \in (0, \pi/m_n - \epsilon_{m_n}).$$

We now choose m_{r+1} such that $(2k_{m_{r+1}}+1)/m_{r+1} < 1/m_r$, $\tau'_{m_{r+1},k_{n_{r+1}}} < \tau_{m_r1}$, and (i_{r+1}) is satisfied.

This m_{r+1} may be chosen so large that (ii_{r+1}) is satisfied when g is extended as the increasing linear mapping of $[(2k_{m_{r+1}}+1)\pi/m_{r+1}, \pi/m_r]$ onto $[b_{r+1}, \tau_{m_r1}]$, where $b_{r+1} \in (\tau'_{m_{r+1},k_{m_{r+1}}}, \tau_{m_r1})$ and is sufficiently close to τ_{m_r1}.

We now extend g as an increasing, continuous, piecewise linear function from $[\pi/m_{r+1}, (2k_{m_{r+1}}+1)\pi/m_{r+1}]$ onto $[\tau_{m_{r+1}1}, b_{r+1}]$ in a manner analogous to the definition of g on $[\pi/m_1, (2k_{m_1}+1)\pi/m_1]$. The slope of g on each interval $[i\pi/m_{r+1}, (i+1)\pi/m_{r+1} - \epsilon_{m_{r+1}}]$ is chosen so small that (iii_{r+1}) holds.

Thus we see that we can define an increasing sequence of integers $\{m_n\}$ with properties (i_n), (ii_n), and (iii_n) for all n. We extend the definition of $g(t)$ to $[0, \pi]$ by setting $g(0)=0$ and we have at once that g is a homeomorphism of $[0, \pi]$ with itself such that

$$\liminf_n \int_0^{\pi} (f \circ g)(t) \frac{\sin m_n t}{t} \, dt \geqq \alpha/\pi.$$

It is relatively easy to extend g to a homeomorphism of $[-\pi, \pi]$ with itself such that

$$\limsup_n \int_{-\pi}^{0} (f \circ g)(t) \frac{\sin m_n t}{t} \, dt \geqq 0.$$

This proves the

THEOREM. *f is such that $f \circ g$ has an everywhere convergent Fourier series for every homeomorphism g if and only if* $\lim_n \alpha_n(\mathcal{I}) = 0$ *for every system $\mathcal{I}$.*

It should be noted that the requirement on a system $\mathcal{I}$ that the I_{nm}, $m=1, \cdots, k_n$, be disjoint is not essential and can be replaced by nonoverlapping, for if we assume that $\mathcal{I}$ satisfies the latter condition and lim $\alpha_n(\mathcal{I})\neq 0$, then by shrinking the intervals of $\mathcal{I}$ slightly we may obtain an $\mathcal{I}'$ satisfying the former condition for which lim $\alpha_n(\mathcal{I}')\neq 0$.

References

1. J. Pál, *Sur des transformations de fonctions qui font converger leurs séries de Fourier*, C.R. Acad. Sci. Paris **158** (1914), 101–103.

2. H. Bohr, *Über einem Satz von J. Pál*, Acta Szeged **7** (1935), 129–135.

3. R. Salem, *Essais sur les séries trigonométrique*, Actualities Sci. Ind. No. 862, Hermann, Paris, 1940.

4. L. C. Young, *Sur une généralisation de la notion de variation de puissance p-ième bornée au sens de M. Wiener, et sur la convergence des séries de Fourier*, C. R. Acad. Sci. Paris **204** (1937), 470–472.

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