ASYMPTOTIC DIOPHANTINE APPROXIMATIONS AND EQUIVALENT NUMBERS

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Let \( \alpha \) be a real irrational number. Define for all real numbers \( B \geq 1 \), \( \lambda(B, \alpha) \) to be the number of solutions in integers \( p, q \) satisfying the inequalities
\[
| q\alpha - p | < \frac{1}{q} \quad \text{and} \quad 1 \leq q \leq B.
\]
The purpose of this note is to consider the following question. If \( \alpha \) and \( \alpha' \) are equivalent irrational numbers (see [2] for all terminology) then is
\[
\lambda(B, \alpha) \sim \lambda(B, \alpha')?
\]
We show that this is usually the case. But there are exceptions, namely when the partial quotients of \( \alpha \) grow too fast.

Let \( \alpha = [a_0, a_1, a_2, \cdots] \) be expressed by means of its continued fraction. Let \( \alpha_n = [a_n, a_{n+1}, \cdots] \) and \( p_n/q_n = [a_0, a_1, \cdots, a_n] \). From [1] we have the following formula for counting:

**Lemma.** Set for all \( v \geq 0 \), \( \lambda_v = \lfloor (\alpha_{v+1} + q_{v-1}/q_v)^{1/2} \rfloor + \rho_v \), where \( \rho_v \) is computed as follows:
- if \( a_{v+1} = 1 \) then \( \rho_v = 0 \),
- if \( a_{v+1} = 2 \) then \( \rho_v = 1 \),
- if \( a_{v+1} \geq 3 \) then \( \rho_v \) has a contribution of 1 for the truth of each of the following:
  1. \( \alpha_{v+1} < q_v/q_{v-1} + 2 \),
  2. \( q_{v+1}/q_v < \alpha_{v+2} + 2 \),

(since \( \rho_v = 0, 1, 2 \)). Then given any \( k, 1 \leq k \leq \lfloor (\alpha_{n+1} + q_{n-1}/q_n)^{1/2} \rfloor \), we have
\[
\lambda(kq_n, \alpha) = \sum_{r=0}^{n-1} \lambda_r + k + O(1) \quad (n \to \infty).
\]
(\([x]\) = largest integer \( \leq x \)).

The main result is expressed in terms of continued fractions. In corollaries we express results for \( \lambda(B, \alpha) \).

**Theorem.** Let \( \alpha = [a_0, a_1, \cdots] \) and \( \alpha' = [a'_0, a'_1, \cdots] \) be equiva-

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Irrational numbers; so say that for \( n \geq n_0 \), \( a_{n+i+l+1} = a'_n \). Let \( \lambda_r, \rho_r \) for \( \alpha \) and \( \lambda'_r, \rho'_r \) for \( \alpha' \) be the quantities of the lemma. Let \( n_0 < n_1 < n_2 < \cdots \) be the integers such that \( \lambda_{n_r+l+1} \neq \lambda'_{n_r}, \quad r = 1, 2, 3, \ldots \)

Then there is a constant \( C > 1 \) such that \( n_r > C^r, \quad r = 1, 2, 3, \ldots \)

Proof. It clearly suffices to assume that \( \alpha = [a_0, \ldots, a_i, \alpha'] \). So suppose \( n > n_0 \) is such that \( \lambda_{n+i+l+1} \neq \lambda'_{n_r} \). Then certainly

\[
[(a_{n+i+l+2} + q_{n+i}/q_{n+i+l+1})^{1/2}] \neq [(a'_{n+i} + q'_{n+i}/q'_{n+i+1})^{1/2}]
\]

or the inequality in (1) or in (2) are different for primed and unprimed symbols. It suffices to consider all the possible cases separately and show that the indices where the given case has inequality increases exponentially. The same type of argument works in all the cases so we assume, say, that

\[
(a_{n+i+l+2} + q_{n+i}/q_{n+i+l+1})^{1/2} > (a'_{n+i} + q'_{n+i}/q'_{n+i+1})^{1/2}
\]

and show these indices increase exponentially.

Now since \( a_{n+i+l+1} = a'_n \) for all \( n \geq 0 \) we have also \( a_{n+i+l+1} = a'_n \). Then in order for (3) to hold we must have \( a_{n+i+l+2} = a'_{n+i} = a^2 - 1 \) for some integer \( a \geq 2 \). Then (3) is equivalent to

\[
a_{n+i+l+3} + q_{n+i}/q_{n+i+l+1} > 1 > a'_{n+i+2} + q'_{n+i}/q'_{n+i+1}.
\]

In particular we have \( q_{n+i}/q_{n+i+l+1} > q'_{n+i}/q'_{n+i+1} \) which is equivalent to \( n \) being even since \( q_{i+2}/q_{i+1} > q'_i/q'_0 \). Further, for the left inequality in (4) to hold, we must have either \( a_{n+i+l+1} = 1 \) or \( a_{n+i+l+3} = 1 \). Again it suffices to show separately that the indices \( n \) where \( a_{n+i+l+3} = 1 \) and (3) holds and where \( a_{n+i+l+1} = 1 \) and (3) holds increase exponentially. Again the arguments are essentially the same so we assume that \( a_{n+i+l+3} = 1 \). Then one checks easily that (4) is equivalent to

\[
q'_{n+i}/q'_{n+i-1} > 1 + a'_{n+i+3} = 1 + a_{n+i+l+4} > q_{n+i+l+1}/q_{n+i}.
\]

So we see that

\[
a_{n+i+l+1} = 1 + a_{n+i+l+4}
\]

and

\[
q'_{n+i-1}/q'_{n+i-2} < a'_{n+i+4} = a_{n+i+l+5} < q_{n+i}/q_{n+i-l-1}.
\]

Hence \( a_{n+i+l} = a_{n+i+l+5} \). Continue and we see that for all \( g, \quad 0 \leq g \leq n-2 \), we have

\[
a_{n+i+l-g} = a_{n+i+l+5+g}
\]

and
\[ \alpha_{2n+l+4} > q_{l+1}/q_l \]

(using the fact that \( n \) is even). To recapitulate we say the sequence of partial quotients of \( \alpha \) has property \( P_n \) for an integer \( n \) provided \( n \) is even, \( a_{n+l+2} = a^2 - 1 \), \( a_{n+l+3} = 1 \) and (5), (6) hold. We will show that the sequence of integers \( n \) such that \( P_n \) is true increases exponentially.

Suppose that \( P_n, P_{n+2r}, P_{n+2s}, 0 < r < s \) are true. We will show that \( s \geq n/10 \). So suppose to the contrary that \( s < n/10 \).

Now let \( t = r \) or \( s \). Then for \( 1 \leq g \leq n - 2 \) we see that

\[ a_{n+l+4+g} = a_{n+l+4+g+4t} \]

by using \( P_n \) and \( P_{n+4t} \), that is the sequence \( a_{n+l+5}, \ldots, a_{2n+l+2} \) is periodic of periods \( 4r \) and \( 4s \). Also the remainder of the sequence \( \text{mod} 4t \) starts the next period. Continue the sequence to be an infinite sequence of period \( 4s \). Then the resulting sequence has period \( 4r \) also; that is for all integers \( x \geq 0 \)

\[ a_{n+l+5+x} = a_{n+l+5+x+4r} \]

To see this we note that it is clear when \( n+l+5+x+4r \leq 2n+l+2 \), that is when \( x \leq n-3-4r \). Since \( 0 < r < s \) and \( 10s < n \) we see that \( 4s \leq n-3-4r \). Thus writing \( x = 4sq+p, \) \( 0 \leq p < 4s \) we have

\[ a_{n+l+5+x} = a_{n+l+5+p} = a_{n+l+5+p+4r} = a_{n+l+5+x+4r} \]

as desired.

So let \( \mu \) be the minimum period of this sequence. Then \( \mu | 4r \) and \( \mu | 4s \). Since \( r < s \) we have \( \mu < 4s \). Singling out some sequence of \( 4s \) terms we have the following type of sequence

\[ b_1 b_2 b_3 b_4 \cdots b_{2s-1} b_{2s} b_{2s+1} b_{2s+2} b_{2s+3} b_{2s+4} \cdots b_{4s-1} b_{4s} \]

where \( x \neq 1 \), which has period \( \mu | 4s, \mu < 4s \). We show that this is impossible, giving the desired contradiction.

Write \( 4s = \rho \mu \) where \( \rho > 1 \). First note that \( 2 | \rho \), since if it did then \( 2s = \rho' \mu \) and

\[ x = b_1 = b_{1+p'\mu} = b_{1+2s} = 1. \]

Thus in particular \( \mu \geq 4 \) and \( \rho \geq 3 \). Now for all \( g \) satisfying \( 2 \leq g \leq 2s-3 \) we have \( b_{2s+1-g} = b_{2s+2+g} \). Choose integers \( \eta, \zeta \) satisfying \( 1 \leq \eta < \zeta \) and \( \eta + \zeta = \rho \). Then one can check that

\[ 2s + 4 \leq 2 + \zeta \mu \leq 4s - 1, \quad 1 \leq 1 + \eta \mu \leq 4s. \]
Thus letting $g = 2 + \xi \mu - (2s + 2) = \xi \mu - 4s$ we have
\[ 1 = b_{2s+1} = b_{2s+2+g} = b_{2s+1-(\mu-2s)} = b_{4s+1-\mu} = b_{1+\mu(\mu-1)} = b_1 = x \]
a contradiction.

The contradiction was obtained by assuming $s < n/10$ so that $s \geq n/10$. Suppose we have $P_{n_1}, P_{n_2}, P_{n_3}, \ldots$. Then with $n = n_r, n+2r = n_{r+1}, n+2s = n_{r+2}$ we have
\[ n_{r+2} - n_r \geq n_r/5 \quad \text{or} \quad n_{r+2} \geq 6n_r/5. \]
Thus we see $n_r \geq C^r$ for some $C > 1$ as desired. This completes the proof of the theorem.

The notation will continue to be the same as that set up in the theorem.

Corollary 1. For all $n$ we have with $1 \leq k \leq \left[ (\alpha_{n+1} + q_{n-1}/q_n)^{1/2} \right]$, $\lambda(kq^\alpha, \alpha') = \lambda(kq_{n+1}, \alpha') + O(\log \log q_n) \ (n \to \infty)$.

Proof. Since $|\lambda_{n+l+1} - \lambda_n| \leq 3$ for any $n$ we have from the theorem and lemma
\[
\lambda(kq^\alpha, \alpha') = \sum_{r=0}^{n-1} \lambda'_r + k + O(1) = \sum_{r=l+1}^{n+l} \lambda_r + k + O(\log n) = \sum_{r=0}^{n+l} \lambda_r + k + O(\log n) = \lambda(kq_{n+l+1}, \alpha) + O(\log n)
\]
and so the result follows since $n = O(\log q_n)$.

The next corollary says, in particular, that for almost all numbers $\alpha, \alpha$ has the same asymptotic estimate as any number equivalent to $\alpha$ (see [3]).

Corollary 2. If $\lambda(B, \alpha) \sim C \log B$ for some constant $C > 0$, and $\alpha'$ is equivalent to $\alpha$, then $\lambda(B, \alpha) \sim \lambda(B, \alpha')$.

Proof. There is a constant $C_1 > 0$ such that $q'_n \leq q_{n+l+1} \leq C_1q_n'$ and so from Corollary 1 we see that $\lambda(kq^\alpha, \alpha') \sim C \log kq_n'$ ($1 \leq k \leq [\alpha_{n+1} + q_{n-1}/q_n]^{1/2}$). Then given $B$ there is an $n$ and such a $k$ with $kq_n' \leq B < (k + 1)q_n'$ or else $[(\alpha_{n+1} + q_{n-1}/q_n)^{1/2}]q_n' \leq B < q_{n+1}$; say $N_1 \leq B < N_2$. Then since $\lambda(N_1, \alpha') = \lambda(N_2, \alpha') + O(1)$ we are done.

Of course, if an explicit error term had been given we could derive one for the difference between $\lambda(B, \alpha)$ and $\lambda(B, \alpha')$.

Corollary 3. There are equivalent numbers $\alpha$ and $\alpha'$ such that $\lambda(B, \alpha)$ and $\lambda(B, \alpha')$ are not asymptotic.
PROOF. It is clear that if \( l \geq 2 \) and \( \alpha = [a_0, \ldots, a_l, \alpha'] \) then there is a \( C_1 > 1 \) (depending only on \( a_0, \ldots, a_l \)) such that \( q_{n+l+1} \leq C_1 q_{n} \). So given \( a_0, \ldots, a_l, l \geq 2 \) we now construct \( \alpha' \). Note first from the Lemma

\[
\lambda(q_n', \alpha') = \sum_{r=1}^{n} a_r'^{1/2} + O(n).
\]

So we may define inductively integers \( k_n, a_{n+1} \) such that

\[
\lim_{n \to \infty} \frac{1}{k_n} \lambda(q_n, \alpha') = 0 \quad \text{and} \quad 1 \leq C_1 k_n < a_{n+1}^{1/2}.
\]

Then using Corollary 1 we see that

\[
\lim_{n \to \infty} \frac{\lambda(k_n q_{n+l+1}, \alpha)}{\lambda(k_n q_{n+l+1}, \alpha')} \leq \lim_{n \to \infty} \frac{\lambda(k_n q_n', \alpha')}{\lambda(C_1 k_n q_n', \alpha')} = \frac{1}{C_1} < 1
\]

as desired.

However we have the following

**Corollary 4.** Let \( \alpha, \alpha' \) be any two equivalent irrationals. Then

\[
\lambda(q_n, \alpha) = \lambda(q_n, \alpha') + O(\log \log q_n) \quad (n \to \infty)
\]

(the error term is smaller than the main term by \([1]\)).

**Proof.** Since \( q_n' \leq q_{n+l+1} \leq C_2 q_n' \) \( (C_2 > 0, \text{ constant}) \) and \( \lambda(k q_n', \alpha') \leq \lambda(q_n', \alpha') + k + O(1) \) for any \( k \) we have by Corollary 1

\[
\lambda(q_{n+l+1}, \alpha) = \lambda(q_n', \alpha') + O(\log \log q_n')
\]

\[= \lambda(q_{n+l+1}, \alpha') + O(\log \log q_{n+l+1}).
\]

**References**