The purpose of this note is to examine the role of nongenerators in the theory of rings, i.e. the elements $x$ of a ring $R$ such that for each subset $M$ of $R$ for which $R = \langle x, M \rangle$, then $\langle M \rangle = R$. The approach used considers a ring as a group with multiple operators and consequently an ideal $A$ generated by a subset $S$ implies that $S \subseteq A$. These results will include those of L. Fuchs [1] and A. Kertesz [2] whenever the ring has unity.

Unless otherwise indicated, the terminology and the necessary known results may be found in N. McCoy's text [3].

Denote the ideal (right ideal) generated by the set $M$ of $R$ by $\langle M \rangle$ (or $\langle M \rangle_r$).

**Definition.** An element $x \in A$ is a generator of an ideal (right ideal) $A$ in a ring $R$ provided that there is a subset $M$ of $A$ such that $A = \langle x, M \rangle$ ($A = \langle x, M \rangle_r$) and $\langle M \rangle \subseteq A$ ($\langle M \rangle_r \subseteq A$) properly. Otherwise $x$ is called a nongenerator of $A$. (Note that $M$ may be empty.)

The set of nongenerators of an ideal (right ideal) $A$ in a ring $R$ will be denoted by $\Phi$ ($\Phi_r$), respectively.

Immediate consequences of the definition are the following:

(i) For an element $x$ of a ring $R$, $x \in \Phi$ ($x \in \Phi_r$) if and only if $\langle x \rangle \subseteq \Phi$ ($\langle x \rangle \subseteq \Phi_r$).

(ii) In a ring $R$, $\Phi$ is an ideal and $\Phi_r$ is a right ideal.

Throughout this paper a maximal ideal of a ring $R$ will be a proper ideal of $R$ that is not contained in another proper ideal of $R$. Similarly for maximal right (left) ideals.

(iii) In a ring $R$, $\Phi$ ($\Phi_r$) is the intersection of the maximal ideals (right ideals), if they exist, and is $R$ otherwise.

(iv) For a ring $R$ and homomorphism $\theta$ of $R$, $\Phi \subseteq \Phi(R\theta)$ and $\Phi_r \subseteq \Phi_r(R\theta)$.

(v) For an ideal $A$ of a ring $R$, $A \subseteq \Phi$ implies that $\Phi(R/A) = \Phi/A$ and $A \subseteq \Phi_r$ implies that $\Phi_r(R/A) = \Phi_r/A$.

(vi) For a ring $R$, if $A$ is an ideal (right ideal) of $R$, then $\Phi(A) \subseteq \Phi$ ($\Phi_r(A) \subseteq \Phi_r$).

(vii) In a ring $R$, $\Phi = (0)$ ($\Phi_r = (0)$) implies that $\Phi(A) = (0)$ ($\Phi_r(A) = (0)$) for each ideal (right ideal) $A$ of $R$. Such rings will be called $\Phi$-free or $\Phi_r$-free respectively.

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(viii) In a ring $R$, $A = \Phi(A)$ ($A = \Phi_r(A)$) for an ideal (right ideal) $A$ of $R$ implies that $A \subseteq \Phi$ ($A \subseteq \Phi_r$).

(ix) If $R$ is a ring and $R = M_1 \oplus \cdots \oplus M_n$, then $\Phi = \Phi(M_1) \oplus \cdots \oplus \Phi(M_n)$ for ideals $M_i$ of $R$.

(x) In a ring $R$, if $A$ is a minimal ideal (right ideal) such that $A \subseteq \Phi$ ($A \subseteq \Phi_r$), then there exists a maximal ideal (right ideal) $B$ such that $R = A \oplus B$.

(xi) If $R$ is a zero ring ($R^2 = (0)$), then $\Phi = \Phi(R^+)$, $\Phi(R^+)$ the Frattini subgroup of the additive group $R^+$.

In the remaining portion of this note, the Jacobson radical and the upper Baer radical will be denoted by $J$ and $N$ respectively.

**Theorem 1.** In a ring $R$, $\Phi \subseteq J$ and $\Phi \subseteq N$.

**Proof.** If $J \neq R$, then $J$ is the intersection of the modular maximal right ideals of $R$; and if $N \neq R$, then $N$ is the intersection of the modular maximal ideals.

Note that in Theorem 1 equality may not occur as the ring \{0, 2; \text{mod } 4\} exemplifies.

**Theorem 2.** In a ring $R$, $RJ \subseteq \Phi$, and $JR \subseteq \Phi_l$. $\Phi_l$ denoting the set of nongenerators with respect to left ideals.

**Proof.** Since the result follows if $\Phi_l = R$, consider the case that $\Phi_l \subseteq R$ properly. Suppose there is an element $x \in R$ such that $yx \in \Phi$, for some element $y \in R$. Then there exists a maximal right ideal $M$ such that $yx \in M$. $M$ defines a simple right $R$-module $R/M \cong R^*$, and under the natural $R$-homomorphism $\theta$ of $R \to R^*$, $y \theta \neq 0$ and $(yx) \theta \neq 0$. So $(R^*)R = R^*$, and an element $z \in R$ exists such that $(yxz) \theta = y \theta$. Then note that if $xz$ is r.q.r., there exists an element $b \in R$ such that $xz + b = xzb$. Under $\theta$, $yxz + yb = yxz + yb \theta$ becomes $(yxz) \theta = -(yb) \theta + (yxb) \theta = -(yb) \theta + (yb) \theta = 0$. So $yxz \in M$ and a contradiction. Therefore $xz$ cannot be r.q.r. In conclusion, if $x$ has the property that $yx \in \Phi$, for some $y \in R$, then $x \in J$. So for each element $x \in J$, $Rx \subseteq \Phi$, i.e. $RJ \subseteq \Phi$. Similarly $JR \subseteq \Phi_l$. (Note: this proof was suggested by a result of Kertesz [2].)

**Corollary 2.1.** (a) For a ring $R$, $\Phi$ and $\Phi_l$ are ideals in $R$.

(b) For a ring $R$, $J^2 \subseteq \Phi_r \cap \Phi_l$.

(c) $J = (\Phi_r : R) = (\Phi_l : R)$

(d) For a ring $R$, $x \in J$ iff $R \times R \subseteq \Phi_r \cap \Phi_l$.

Since in general both $\Phi_r$ and $\Phi_l$ are in $J$, it follows that in each primitive ring the right ideals and the left ideals are $\Phi_r$- and $\Phi_l$-free respectively. If the ring is a simple nonradical ring, then the ring is
\(\Phi\)-free. For the simple primitive rings, all three hold. And for a field \(F\), \(\Phi(F) = (0)\).

In general \(\Phi \subseteq J\). For example: let \(R\) be the ring of all linear transformations of a vector space \(V\) with a denumerable basis. It is known (e.g., see [3]) that \(R\) is a primitive ring and \(J = (0)\). Since \(R\) has unity, \(N = R\); and, in fact, the only proper ideal besides \((0)\) is the ideal of elements of finite rank. This ideal is \(N = \Phi\). Also note that \(\Phi, = \Phi_1 = (0)\).

**Theorem 3.** For a ring \(R\) having \(R^2 = R\), \(\Phi\) is a semiprime ideal.

**Proof.** Each maximal ideal is prime. If \(A\) is an ideal for which \(A^2 \subseteq \Phi\), then \(A^2\) is contained in each maximal ideal \(M\). So \(A\) is contained in each \(M\). Therefore \(A \subseteq \Phi\).

**Corollary 3.1.** For a ring \(R\) having \(R^2 = R\), the prime radical is contained in \(\Phi\).

**Corollary 3.2.** For a ring \(R\) having \(R^2 = R\), \(J \subseteq \Phi\) iff \(J^2 \subseteq \Phi\).

**Theorem 4.** For a ring \(R\) having \(R^2 = R\) and center \(Z\), \(N \cap Z \subseteq \Phi\), and \(N \cap Z \subseteq \Phi\).

**Proof.** If \(A = N \cap Z \subseteq \Phi\), and \(M\) is a maximal right ideal not containing \(A\), then \(R = A + M\). This implies that \(M\) is a maximal ideal, and \(R^2 = R\) implies that \(R / M\) is a simple commutative nonzero ring. Hence \(M\) is modular and \(N \subseteq M\) implies that \(A \subseteq M\). So \(A \subseteq \Phi\). Similarly \(N \cap Z \subseteq \Phi\).

**Corollary 4.1.** If \(R\) is a commutative ring and \(R^2 = R\), then \(J = \Phi\).

**Theorem 5.** For a ring \(R\) having \(R^2 = R\), \(\Phi, = \Phi_1 = J\).

**Proof.** Consider \(\Phi,\) and note that for \(R = \Phi,\) and \(\Phi, \subseteq J\) implies that \(J = \Phi,\). So then consider the case that \(\Phi, \subseteq R\) properly. By Theorem 2, \(J^2 \subseteq \Phi,\). Form \(R / J^2 \cong R^*\) noting that \(J^* \cong J(R/J^2) = J / J^2\) and that \(\Phi, \cong \Phi, (R/J^2) = \Phi, / J^2\). If \(x \in J^*\) and \(x \in \Phi^*\), there exists a maximal right ideal \(M^*\) such that \(x \in M^*\). Under the natural \(R^*-\)homomorphism \(\theta\) of \(R^* \rightarrow R^* / M^*\), \(R^*\) is mapped onto a simple right \(R^*-\)module \(R^* / M^*\). Since \(x \in M^*\), then \(J^* \theta = R^* / M^*\). But \(J^* \theta = (0)\) implies that \(R^* / M^*\) is annihilated by \(R^*\), i.e. \((R^* / M^*)R^* = (0)\). This contradicts the hypothesis that \(R^2 = R\) since, in turn, this implies that \(R^{*2} = R^*\) and \((R^* / M^*)R^* = R^* / M^*\). So \(J^* \subseteq M^*\). This leads to \(J^* \subseteq \Phi^*\) and hence \(J \subseteq \Phi^*\). So the result follows.

Similarly \(\Phi_1 = J\).

**Corollary 5.1 (L. Fuchs [1]).** For a ring with unity, \(\Phi, = \Phi_1 = J\).
Theorem 6. If \( R \) satisfies the d.c.c. on right ideals, then \( \Phi = (0) \) if and only if \( R \) is a direct sum of a finite collection of simple ideals.

Proof. Consider the intersections of all finite collections of maximal ideals. By the d.c.c. on right ideals, each linear system has a minimal element, say \( D \). If \( M \) is a maximal ideal, then \( D = D \cap M \). So \( D \subseteq \Phi \) and \( D = (0) \). As is known, if there exists in a ring a finite number of maximal ideals \( M_i \), \( i = 1, \ldots , n \) with zero intersection, then \( R \) is isomorphic to the direct sum of some or all the simple rings \( R/M_i \), \( i = 1, \ldots , n \). By (ix) each direct summand has \( \Phi(R/M_i) = (0) \) since \( R/M_i \) is a simple ideal.

Again by (ix) the converse is evident.

Theorem 7. If \( R \) is a ring with d.c.c. on right ideals, then both \( \Phi \) and \( \Phi_1 \) are contained in \( \Phi \).

Proof. The theorem is valid whenever \( R = \Phi \), so consider the case that \( \Phi \subset R \) properly. In particular restrict attention to \( R^* = R/M \) for a maximal ideal \( M \). For either \( R^* = (0) \) or \( R^* = R^* \), \( \Phi^* = (0) \). Hence under the natural homomorphism \( \theta \) of \( R \rightarrow R^* \), \( \Phi \theta \subseteq (0) \) implies that \( \Phi_r \subseteq M \). So \( \Phi_r \subseteq \Phi \) and similarly \( \Phi_1 \subseteq \Phi \).

Theorem 8. If \( R \) is a ring with the d.c.c. on right ideals, then \( \Phi_r = \Phi_1 = \Phi \).

Proof. Since \( \Phi_r \) is an ideal of \( R \) form \( R^* \cong R/\Phi_r \) having \( \Phi^* = \Phi_r(R^*) = (0) \), \( \Phi^* \cong \Phi_r/\Phi_r \), and \( J^* \cong J/\Phi_r \). If \( M^* \) is a maximal right ideal such that \( \Phi^* \subseteq M^* \), then \( R^* = \Phi^* + M^* \). However, since \( R^* J^* \subseteq \Phi^* = (0) \), then \( \Phi^* \) is in the annihilator of \( M^* \). This implies that \( M^* \) is an ideal of \( R^* \) and hence a contradiction to the assumption that \( \Phi^* \subseteq M^* \). So \( \Phi^* = (0) \), i.e. \( \Phi \subseteq \Phi_r \), and the result follows. Similarly, \( \Phi_1 = \Phi \).

Theorem 9. For a ring \( R \) with d.c.c. on right ideals and \( R \) not a radical ring, then \( \Phi = J \) if and only if \( R^2 = R \).

Proof. Suppose \( R^2 = R \) and there exists a maximal ideal \( M \) such that \( J \subseteq M \). Then under the natural homomorphism \( \theta \) of \( R \rightarrow R/M = R^* \), \( J \theta = R^* \). However, since \( J^2 \subseteq \Phi \subseteq M \) it follows that \( R^* = (0) \) and this contradicts \( R^2 = R \). So \( J \subseteq \Phi \). Since \( J = N \) and \( \Phi \subseteq N \), then \( \Phi = J \).

On the other hand, suppose that \( J \subseteq \Phi \subseteq R \) properly. Form \( R/\Phi \cong R^* \) and note that \( J^* \cong J(R/\Phi) = (0) = \Phi^* \cong \Phi(R/\Phi) \). As is known, \( J^* = (0) \) implies that \( R^* = R^* \). If \( R^2 \subseteq R \) properly and \( R^2 \theta = R^* \) under the natural homomorphism \( \theta \) of \( R \rightarrow R^* \), then \( R = \Phi + R^2 = R^2 \) and a contradiction. So \( R^2 = R \).
In a radical ring $R$ the condition $\Phi = J$ does not necessarily imply that $R^2 = R$. For example, let $R$ be a zero ring having $R^+$ a group of type $p^\infty$. Then $\Phi(R) = \Phi(R^+) = R^+$ and $J = R$.

**Bibliography**


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