NTH ROOTS OF OPERATORS

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I. **Introduction.** Let A and E be continuous linear transformations which map a Banach space into itself. If E commutes with A^n, for some integer n greater than 1, we do not necessarily know whether E commutes with A. The main purpose of this paper is to exhibit a proof of the following theorem:

Let A and E be continuous linear transformations which map a complex Banach space into itself. Let n be an integer greater than 1. *If* $\sigma(A) \cap \sigma(e^{2\pi ik/n}A) = \emptyset$ *for* $k=1, \cdots, n-1$ *and E commutes with A^n, then E commutes with A.*

A significant corollary of this theorem is as follows:

If A is a continuous linear transformation from a Hilbert space into itself and A^n is normal, then A is normal, whenever the spectrum of A satisfies the condition stated in the theorem.

Throughout the paper the following notation and terminology will be used. $\mathfrak{X}$ is a complex Banach space and $\mathfrak{L}(\mathfrak{X})$ is the space of continuous linear transformations from $\mathfrak{X}$ into $\mathfrak{X}$. The identity element of $\mathfrak{L}(\mathfrak{X})$ will be denoted by I. If $A \in \mathfrak{L}(\mathfrak{X})$, $\sigma(A)$ will denote the spectrum of A and $\rho(A)$ the resolvent set of A. If C is a simple closed rectifiable curve in the complex plane, C will be assumed to be oriented in a counterclockwise direction. By the interior of C we mean the bounded component of the complement of C; the exterior of C is the unbounded component.

II. **Essential background material.** The proof of the above-stated theorem relies heavily on the following results, which are found in *The spectrum of linear transformations* by E. R. Lorch [**1**]:

THEOREM 1. *Let C be a simple closed rectifiable curve lying entirely within $\rho(A)$, where $A \in \mathfrak{L}(\mathfrak{X})$. Then the contour integral*

$$P = \frac{1}{2\pi i}\int_C (zI - A)^{-1}dz$$

exists and represents an element of $\mathfrak{L}(\mathfrak{X})$.

The transformation P is unchanged if the curve C is continuously

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deformed into a curve C', providing only that the deformation is effected without going outside of $\rho(A)$ [**1**, p. 241].

THEOREM 2. *The operator P defined by the curve C as in Theorem* 1 *has the following properties*:

(a) $P=0$ *if and only if every point in the interior of C lies in* $\rho(A)$.

(b) $P=I$ *if and only if every point in the exterior of C lies in* $\rho(A)$ [**1**, p. 244].

III. **Basic lemmas.** It is well known that if $A\in\mathfrak{L}(\mathfrak{X})$, $\sigma(A)$ is a nonempty, compact subset of the complex plane. If for some integer n, $n>1$, $\sigma(A)\cap\sigma(e^{2\pi ik/n}A)=\varnothing$ for $k=1,\cdots,n-1$, then $M>0$, where

$$M = \min_{k=1,\cdots,n-1} \text{g.l.b.}\{\,|z-w| : z\in\sigma(A), w\in\sigma(e^{2\pi ik/n}A)\}.$$

Since $\sigma(A)$ is compact, there exist $z_1,\cdots,z_m$ in $\sigma(A)$ such that $\sigma(A)\subset\bigcup_{j=1}^m B(z_j;\, M/3)$, where $B(z;\, M/3)$ is the open ball in the complex plane of radius $M/3$ and center z. Let $V_1,\cdots,V_p$ be the components of $\bigcup_{j=1}^m B(z_j;\, M/3)$ and let

$$s_{p(k-1)+j} = \sigma(e^{2\pi ik/n}A)\cap(e^{2\pi ik/n}V_j)$$

for $k=1,\cdots,n$ and $j=1,\cdots,p$. Note the following facts:

(1) for each j, s_j is a compact subset of the complex plane;

(2) for $q\neq j$, $s_q\cap s_j=\varnothing$;

(3) for $k=1,\cdots,n$, $\sigma(e^{2\pi ik/n}A)=\bigcup_{j=1}^p s_{p(k-1)+j}$.

Let $S=\{s_k: k=1,\cdots,np\}$. We will say that s_j *is interior to* s_q if $j\neq q$ and whenever C is a simple closed rectifiable curve such that s_q is in the interior of C, then s_j is also in the interior of C. We will denote this by $s_j<s_q$. Because of the way in which S was constructed, if $s_j<s_q$, there exists a simple closed rectifiable curve C' such that s_j is in the interior of C' and s_q is in the exterior of C'.

C will be called a *proper curve for* s_j if C is a simple closed rectifiable curve, lying entirely in $\bigcap_{k=1}^n \rho(e^{2\pi ik/n}A)$, such that (1) s_j is in the interior of C and (2) whenever s_q is in the interior of C, then $s_q=s_j$ or $s_q<s_j$.

An element s_k of S will be said to be of *order* 0 if there exists no s_j in S such that $s_j<s_k$. Since S is a finite set, there must exist elements of S of order 0. A *branch of length* k in S is a sequence $\{t_j: j=1,\cdots,k\}$ of elements of S such that (1) t_1 is of order 0, (2) $t_j<t_{j+1}$ for $j=1,\cdots,k-1$, and (3) there exists no s_i in S such that $t_j<s_i<t_{j+1}$. An element s_i of S is of *order* k, $k\geqq 1$, if there exists a branch $\{t_j: j=1,\cdots,k\}$ of length k in S such that $t_k<s_i$ and there

exists no branch $\{\bar{t}_j: j=1, \cdots, k+1\}$ of length $k+1$ in S such that $\bar{t}_{k+1} < s_i$.

LEMMA 1. *Let n be a positive integer. There exist nonzero complex numbers $b_1, \cdots, b_n$ such that*

$$(z^n I - A^n)^{-1} = A^{-(n-1)} \sum_{j=1}^{n} b_j (zI - e^{2\pi ij/n} A)^{-1},$$

whenever A is an invertible element of $\mathfrak{L}(\mathfrak{X})$ and $z \in \bigcap_{j=1}^{n} \rho(e^{2\pi ij/n} A)$.

PROOF. Let $\{b_1, \cdots, b_n\}$ be the unique solution of the system of equations

$$\sum_{k=1}^{n} b_k e^{2\pi ik(j-1)/n} = 0 \quad \text{for } j = 1, \cdots, n-1,$$

$$\sum_{k=1}^{n} b_k e^{2\pi ik(n-1)/n} = 1.$$

Let z be any element of $\bigcap_{j=1}^{n} \rho(e^{2\pi ij/n} A)$. Then

$$\begin{aligned} A^{n-1} &= \sum_{j=1}^{n} z^{n-j} A^{j-1} \left(\sum_{k=1}^{n} b_k e^{2\pi ik(j-1)/n} \right) \\ &= \sum_{k=1}^{n} b_k \left(\sum_{j=1}^{n} z^{n-j} (e^{2\pi ik/n} A)^{j-1} \right) \\ &= \sum_{k=1}^{n} b_k (zI - e^{2\pi ik/n} A)^{-1} (z^n I - A^n) \\ &= (z^n I - A^n) \sum_{k=1}^{n} b_k (zI - e^{2\pi ik/n} A)^{-1}. \end{aligned}$$

Therefore, $(z^n I - A^n)^{-1} = A^{-(n-1)} \sum_{k=1}^{n} b_k (zI - e^{2\pi ik/n} A)^{-1}$.

To see that each of the b_j is nonzero we need only consider a special case. Let $A = I$. Let C be a simple closed rectifiable curve such that $e^{2\pi ij/n}$ is in the interior of C and $e^{2\pi ik/n}$, $k \neq j$, is in the exterior of C. Then, using the last equation in the preceding paragraph, we see that

$$\begin{aligned} \frac{1}{2\pi i} \int_C (z^n - 1)^{-1} dz &= \sum_{k=1}^{n} b_k \frac{1}{2\pi i} \int_C (z - e^{2\pi ik/n})^{-1} dz \\ &= b_j. \end{aligned}$$

However,

$$\frac{1}{2\pi i} \int_C (z^n - 1)^{-1} dz = \prod_{k=1; k \neq j}^{n} (e^{2\pi ij/n} - e^{2\pi ik/n})^{-1} \neq 0.$$

Therefore, $b_j \neq 0$, $j = 1, \cdots, n$.

The most serious problem which arose in the proof of Theorem 3 was the interrelation between the components of $\sigma(A)$ and the components of $\sigma(e^{2\pi ik/n}A)$, $k=1, \cdots, n-1$. If there exists a simple closed rectifiable curve C such that $\sigma(A)$ is interior to C and $\sigma(e^{2\pi ik/n}A)$ is exterior to C for $k=1, \cdots, n-1$, then by Theorem 2 we have

$$\int_C (zI - e^{2\pi ik/n}A)^{-1}dz = 0 \quad \text{for } k = 1, \cdots, n-1$$

and $(1/2\pi i)\int_C (zI-A)^{-1}dz=I$. Using Lemma 1, we see that

$$\frac{1}{2\pi i}\int_C (z^nI - A^n)^{-1}dz = A^{-(n-1)}b_n.$$

Therefore, if E commutes with A^n, E also commutes with $A^{-(n-1)}$ and with A. However, in the case in which there exists no such curve C, the situation is much more complicated.

LEMMA 2. *Let A and E be elements of $\mathfrak{L}(\mathfrak{X})$ and let n be an integer greater than* 1. *Suppose that*

$$\sigma(A) \cap \sigma(e^{2\pi ik/n}A) = \emptyset \quad \text{for } k = 1, \cdots, n-1$$

and that E commutes with A^n. If s_j is an element of S and C is a proper curve for s_j, then E commutes with

$$A^{-(n-1)}\int_C (zI - e^{2\pi ik/n}A)^{-1}dz, \quad k = 1, \cdots, n.$$

PROOF. The proof of this lemma will be by induction on the order of the elements of S. Note that if C is any simple closed rectifiable curve, lying in $\bigcap_{k=1}^n \rho(e^{2\pi ik/n}A)$, then $\int_C (z^nI-A^n)^{-1}dz$ is an element of $\mathfrak{L}(\mathfrak{X})$ and commutes with E [1, pp. 240–241].

Let s_j be an element of S of order 0 and let C be a proper curve for s_j. Assume that $s_j \subset \sigma(e^{2\pi ik/n}A)$. By Theorem 2,

$$\int_C (zI - e^{2\pi iq/n}A)^{-1}dz = 0 \quad \text{for } q \neq k \quad \text{and} \quad q = 1, \cdots, n. \tag{1}$$

Using (1) and Lemma 1, we have

$$\int_C (z^nI - A^n)^{-1}dz = b_kA^{-(n-1)}\int_C (zI - e^{2\pi ik/n}A)^{-1}dz. \tag{2}$$

From (1) and (2) it follows that E commutes with

$$A^{-(n-1)}\int_C (zI - e^{2\pi iq/n}A)^{-1}dz \quad \text{for } q = 1, \cdots, n$$

whenever C is a proper curve for an element of S of order 0.

Assume now that whenever s_j is of order p, $0 \leqq p \leqq m$, and C is a proper curve for s_j, E commutes with

$$A^{-(n-1)} \int_C (zI - e^{2\pi iq/n}A)^{-1}dz \quad \text{for } q = 1, \cdots, n.$$

Let s_{j_0} be an element of S of order $m+1$ and C be a proper curve for s_{j_0}. Let $t_1, \cdots, t_r$ be the elements of S such that $t_j < s_{j_0}$, $j=1, \cdots, r$ and there exists no s_k in S such that $t_j < s_k < s_{j_0}$. Let C_j, $j=1, \cdots, r$, be a proper curve for t_j. Since each of the t_j, $j=1, \cdots, r$, is of order p, $p \leqq m$, then E commutes with

$$A^{-(n-1)} \int_{C_j} (zI - e^{2\pi ik/n}A)^{-1}dz \quad \text{for } k = 1, \cdots, n \text{ and } j = 1, \cdots, r.$$

Assume that $s_{j_0} \subset \sigma(e^{2\pi ik/n}A)$. It follows from Theorems 1 and 2 that for $q \neq k$,

$$A^{-(n-1)} \int_C (zI - e^{2\pi iq/n}A)^{-1}dz = \sum_{p=1}^{r} A^{-(n-1)} \int_{C_p} (zI - e^{2\pi iq/n}A)^{-1}dz.$$

Therefore, the induction hypothesis implies that E commutes with

$$A^{-(n-1)} \int_C (zI - e^{2\pi iq/n}A)^{-1}dz \quad \text{for } q \neq k \quad \text{and} \quad q = 1, \cdots, n.$$

Moreover, since E commutes with $\int_C (z^nI - A^n)^{-1}dz$ and

$$\int_C (z^nI - A^n)^{-1}dz = \sum_{q=1}^{n} b_q A^{-(n-1)} \int_C (zI - e^{2\pi iq/n})^{-1}dz,$$

E also commutes with $A^{-(n-1)} \int_C (zI - e^{2\pi ik/n}A)^{-1}dz$. Thus the proof by induction is complete.

IV. Theorems.

THEOREM 3. *Let A and E be elements of $\mathfrak{L}(\mathfrak{X})$ and let n be an integer greater than* 1. *If $\sigma(A) \cap \sigma(e^{2\pi ik/n}A) = \varnothing$ for $k=1, \cdots, n-1$, and A^n commutes with E, then A commutes with E.*

PROOF. Let $t_1, \cdots, t_r$ be the elements of S which are interior to no other elements of S. Let C_j, $j=1, \cdots, r$, be a proper curve for t_j. Let C be a simple closed rectifiable curve such that t_j is interior to C for $j=1, \cdots, r$. By Theorem 2

$$\frac{1}{2\pi i} \int_C (zI - A)^{-1}dz = I \tag{1}$$

since $\sigma(A)$ lies entirely in the interior of C. Moreover, by Theorem 1 and Theorem 2

$$\frac{1}{2\pi i}\int_C (zI - A)^{-1}dz = \sum_{j=1}^{r} \frac{1}{2\pi i}\int_{C_j} (zI - A)^{-1}dz. \tag{2}$$

Combining (1) and (2), we see that

$$\sum_{j=1}^{r} \frac{A^{-(n-1)}}{2\pi i}\int_{C_j} (zI - A)^{-1}dz = A^{-(n-1)}. \tag{3}$$

Lemma 2 asserts that E commutes with each of the summands in the left hand side of equation (3). Therefore E commutes with $A^{-(n-1)}$ as well as A^n, which implies that E commutes with A.

One of the simplest situations in which the hypotheses of Theorem 3 are satisfied is when $\mathfrak{X}$ is a Hilbert space and either $\operatorname{Re} A \gg p > 0$ or $\operatorname{Im} A \gg p > 0$, for some real number p. Then A and A^2 commute with exactly the same operators, since $\sigma(A) \cap \sigma(-A) = \emptyset$. If in addition A^2 is normal, $AA^{*2} = A^{*2}A$ since $A^2A^{*2} = A^{*2}A^2$. But this implies that $AA^* = A^*A$ since $\sigma(A^*) \cap (-A^*) = \emptyset$. More generally, we have by an analogous proof:

THEOREM 4. *Let $\mathfrak{X}$ be a Hilbert space. If A is an element of $\mathfrak{L}(\mathfrak{X})$ such that A^n is normal for some n greater than* 1 *and $\sigma(A) \cap \sigma(e^{2\pi ik/n}A) = \emptyset$ for $k = 1, \cdots, n-1$, then A is normal.*

REFERENCE

1. E. R. Lorch, *The spectrum of linear transformations*, Trans. Amer. Math. Soc. 52 (1942), 238–248.

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