SOME COMMENTS ON THE STRUCTURE OF COMPACT DECOMPOSITIONS OF S^3

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In this note we derive some theorems of a general nature on compact, upper semicontinuous decompositions of S^3, spherical 3-space. A compact decomposition G of S^3 is one obtained from a compact, proper subset D of S^3 by setting $G=\{g: g \text{ is either a component of } D \text{ or a point of } S^3-D\}$. In this paper a point-like, compact decomposition of S^3 is one such that the complement of each component of D is homeomorphic to E^3, Euclidean 3-space; and a 1-dimensional, compact decomposition of S^3 is one such that each component of D has dimension no greater than 1.

Theorem 1 shows that in the decomposition space of a point-like, compact decomposition of S^3 the collection of points which fail to have some neighborhood homeomorphic to E^3 is dense in itself. The remaining theorems are concerned with the effect of inserting or removing certain elements from a compact decomposition of S^3. Armentrout's [2, Theorems 1, 6] are essential to our investigation. A version of these theorems appears below as Theorem A. A summary of some known results on compact decompositions of S^3 may be found in [3].

We assume then that G denotes a compact decomposition of S^3 with associated projection map P onto the decomposition space S^3/G. Let H_G denote the sum of the nondegenerate elements of G. Note that $P(\mathrm{Cl}\, H_G)$ is compact and 0-dimensional. It is known that there is a sequence of compact polyhedral 3-manifolds with boundary in S^3 such that $\bigcap_{i=1}^{\infty} M_i=\mathrm{Cl}\, H_G$ and, for each i, $M_{i+1}\subset \mathrm{Int}\, M_i$. Such a sequence M_i will be referred to as a defining sequence for G. Let $Q=\{x: x\in S^3/G \text{ and } x \text{ has a neighborhood homeomorphic to } E^3\}$, and let $F=S^3/G-Q$. Note that if $F\neq\emptyset$, then F is compact and 0-dimensional.

THEOREM A. *Assume G is a compact decomposition of S^3 ($P(\mathrm{Cl}\, H_G)$ is compact and 0-dimensional). Let M be a compact polyhedral 3-manifold with boundary in S^3 such that* $\mathrm{Bd}\, M\cap \mathrm{Cl}\, H_G=\emptyset$. *If $P(M)\subset Q$ and either G is a point-like decomposition or M has a triangulation whose 1-skeleton is disjoint from* $\mathrm{Cl}\, H_G$, *then there is a map of M onto itself fixed on* $\mathrm{Bd}\, M$ *and inducing the same decomposition as G restricted to M.*

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PROOF. Since either G is a point-like decomposition or M has a triangulation whose 1-skeleton is disjoint from Cl H_G, it follows that there is a collection of disjoint polyhedral arcs $A_1, \cdots, A_n$ such that each Int $A_i \subset$ Int $M -$ Cl H_G, each Bd $A_i \subset$ Bd M, and any two boundary components of any component of M are connected by one of the A_i. For each i, let A_i' be an open regular neighborhood of A_i in M such that Cl $A_i' \cap$ Cl $A_j' = \emptyset$ for $i \neq j$ and each Cl A_i' $\cap$ Cl $H_G = \emptyset$. Let $M' = M - \bigcup_{i=1}^n A_i'$. Since the boundary of each component of M' is connected, it follows from the proof of either [2, Theorem 1] (if G is point-like) or [2, Theorem 6] (if M has a triangulation whose 1-skeleton is disjoint from Cl H_G) that there is a homeomorphism h' of M' onto $P(M')$ such that $h' | \text{Bd } M' = P | \text{Bd } M'$. Extending h' by using P on each A_i', we see that there is a homeomorphism h of M onto $P(M)$ such that $h | \text{Bd } M = P | \text{Bd} M$. The required map is $h^{-1}P$.

THEOREM 1. *Let G be a point-like, compact decomposition of S^3. Then F is either empty or a Cantor set, that is F has no isolated points.*

PROOF. Suppose that F has an isolated point x. In the defining sequence M_i of G, let n be chosen large enough that the component K of M_n containing $P^{-1}(x)$ is such that $P(K) - x \subset Q$. Since G is point-like there is a map f' of K onto itself such that f' is fixed on Bd K, f' is a homeomorphism on $K - P^{-1}(x)$ and $f'(P^{-1}(x))$ is a point. For $i = 1, 2, 3, \cdots$, let $K_i = M_{n+i} \cap K$ and let L_i be the component of K_i containing $P^{-1}(x)$. Let $K_1' = K_1 - L_1$, and, for each i, let $K_{i+1}' = K_{i+1} - \bigcup_{j=1}^{i}(K_j') \cup L_{i+1}$. Since each $P(K_i') \subset Q$, it follows from Theorem A that there is a mapping f_i of K_i' onto itself fixed on Bd K_i' and inducing the same decomposition as G restricted to K_i'. Assuming each $f_i =$ identity on $K - K_i'$, we have that $f = f'(\prod_{i=1}^{\infty} f_i)$ is a map of K onto itself fixed on Bd K and inducing the same decomposition as G restricted to K. It follows that the map Pf^{-1} is a homeomorphism of K onto $P(K)$. But this contradicts that $x \in F$. Therefore F is either empty or it is homeomorphic to a Cantor set.

In [6], Finney showed that if f is a point-like, simplicial map of S^3, then $f(S^3)$ is homeomorphic to S^3. To prove this he simplified the decomposition G_f of S^3 induced by f. This simplification was accomplished by a process of deleting portions of certain nondegenerate elements of G_f to obtain a new decomposition G_f' such that S^3/G_f' is homeomorphic to S^3/G_f. In Theorems 2 and 3 it is shown that, for compact decompositions G of S^3, we may delete certain elements of G to form a new decomposition G' such that S^3/G is homeomorphic to S^3/G'.

THEOREM 2. *Let G be a point-like, compact decomposition of S^3. Let U be an open set in S^3 such that U is the union of elements in G and $P(U)\subset Q$. Let G' be the decomposition obtained by points of U and elements of G in S^3-U. Then S^3/G is homeomorphic to S^3/G'.*

PROOF. It is easily checked that G' is an upper semicontinuous decomposition of S^3. Let P' be the projection map for G'. Let M_i be a defining sequence for G. Define K_i to be the union of all components of M_i which do not intersect S^3-U. Let $K_1'=K_1$ and, for $i=1, 2, \cdots$, let $K_{i+1}'=K_{i+1}-\bigcup_{j=1}^{i} K_j'$. Since $P(K_i')\subset Q$ it follows by Theorem A that there is a map f_i of S^3 onto itself fixed on $S^3-\text{Int } K_i'$ and inducing the same decomposition as G restricted to K_i'. Let $f=\prod_{i=1}^{\infty} f_i$. Then $P'fP^{-1}$ is a 1-1 correspondence between the points of S^3/G and S^3/G'. If K is a component of some K_i', then $f^{-1}(K)=K$. Using this fact and that f is a continuous map on U that induces the same decomposition as G restricted to U, it follows that if V is open in S^3 and V is the union of elements in G', then $f^{-1}(V)$ is open in S^3 and $P(f^{-1}(V))$ is open in S^3/G. Hence $P'fP^{-1}$ is a continuous 1-1 map of S^3/G onto S^3/G', and it follows that $P'fP^{-1}$ is a homeomorphism between these spaces.

Theorem 3 is similar to Theorem 2, except that we consider 1-dimensional, compact decompositions of S^3 instead of point-like, compact decompositions of S^3. To prove Theorem 3 the following lemma is needed. A version of this lemma was proved independently by Alford and Sher in [1], but our proof differs from theirs in that we do not use Kwun and Raymond's [7, Theorem 3, Corollary 2].

LEMMA. *Let G be a* 1*-dimensional, compact decomposition of S^3 and let M be a compact polyhedral* 3*-manifold with boundary in S^3 such that* Bd $M\cap$Cl $H_G=\emptyset$, *$P(M)\subset Q$, and $P(M)$ is embeddable in S^3. Then there is a map of M onto itself fixed on* Bd M *and inducing the same decomposition as G restricted to M.*

PROOF. Let T be a triangulation of M such that each 1-simplex of T that intersects Bd M is disjoint from Cl $H_G\cap M$. Since dim $g\leqq 1$ for each $g\in G$, we may assume that the 0-skeleton of T is disjoint from Cl H_G. Let A be a 1-simplex of T with endpoints a and b such that $A\cap$Bd $M=\emptyset$, and let K be a polyhedral cube in Int M obtained by thickening Int A slightly so that Bd $A\subset$Bd K and $K\cap T_1=A$, where T_1 is the carrier of the 1-skeleton of T. Let U be a complementary domain of Bd K. Since dim $g\leqq 1$ for each $g\in G$ and $P($Cl $H_G)$ is 0-dimensional, it follows that $U-$Cl H_G is connected. Hence there is an arc X with endpoints a and b such that Int $X\subset U$ and $X\cap$Cl H_G

$=\emptyset$. Similarly there is an arc Y with endpoints a and b such that Int $Y\subset(S^3-\text{Cl } U)$ and $Y\cap\text{Cl } H_G=\emptyset$. Since Bd $M\cap\text{Cl } H_G=\emptyset$, we may assume $X\cup Y\subset\text{Int } M$.

Suppose Bd $K\cap\text{Cl } H_G$ separates a from b on Bd K. Let S be the component of Bd $K-\text{Cl } H_G$ containing a. It follows that $P(\text{Cl } S)$ is a singular 2-sphere in Int $P(M)$ (regard $P(\text{Cl } S)$ as the image of Bd K under the map $g(x)=P(x)$ if $x\in S$ and $g(x)=P(\text{Bd } C)$ if x belongs to the component C of Bd $K-S$) and that the simple closed curve $P(X\cup Y)$ intersects and pierces $P(\text{Cl } S)$ at just one point. But this is impossible since $P(M)$ is embeddable in S^3. Hence there is an arc B on Bd K with endpoints a and b such that $B\cap\text{Cl } H_G=\emptyset$. There is a homeomorphism of M onto itself which is fixed on (Bd $M\cup T_1$) $-$Int A and takes A onto B. Repeating this argument a finite number of times, we push T_1 off Cl H_G. Since $P(M)\subset Q$, this lemma now follows from Theorem A.

THEOREM 3. *Let G be a 1-dimensional, compact decomposition of S^3 and let U be an open set in S^3 such that U is the union of elements in G and $P(U)\subset Q$. Let G' be the decomposition obtained by points of U and elements of G in S^3-U. Then S^3/G is homeomorphic to S^3/G'.*

PROOF. As in Theorem 2 we may choose a sequence K_i' of disjoint 3-manifolds in U such that $U\cap\text{Cl } H_G\subset\bigcup_{i=1}^{\infty}K_i'$. Each K_i' can be chosen so that $P(K_i')$ is embeddable in S^3. By the previous lemma there is a map of S^3 onto itself fixed on $S^3-\text{Int } K_i'$ and inducing the same decomposition as G restricted to K_i'. The remainder of this proof is the same as the proof of Theorem 2.

COROLLARY. *Let G be a 1-dimensional, compact decomposition of S^3 such that S^3/G is a 3-manifold. Then S^3/G is homeomorphic to S^3.*

As pointed out by Bing in [**4**, p. 7], it may be shown that if C_1, $C_2, \cdots, C_n$ are mutually exclusive, nonseparating continua in S^3, then there is a decomposition G of S^3 such that each $C_i\in G$ and S^3/G is homeomorphic to S^3. The next theorem shows that while staying in the category of point-like, compact decompositions we cannot change an element of G whose image is in F to one whose image is in Q by adding more nondegenerate elements to G.

THEOREM 4. *Let G and G' be point-like, compact decompositions of S^3 such that each nondegenerate element of G is contained in G'. If $g\in G$ and $P(g)\in F$, then $P'(g)\in F'$ (where P' is the projection map associated with G' and F' is the non-Euclidean points of S^3/G').*

PROOF. Assume by way of contradiction that g_0 is an element of G such that $P(g_0) \in F$ but $P'(g_0) \in Q'$ (where $Q' = S^3/G' - F'$). Let M_i and M_i' be defining sequences for G and G', respectively. Let K be a component of some M_n such that $g_0 \subset K$, $P'(K) \subset Q'$, and, for each $i = 1, 2, \cdots$, let $K_i = K \cap M_{n+i}$. If for each positive integer m and each positive number ϵ there is a homeomorphism h of S^3 onto itself such that: (1) if $x \in S^3 - K_m$, then $h(x) = x$; and (2) if $g \in G$ and $g \subset K_m$, then diam $h(g) < \epsilon$, then it follows from the proof of Theorem 1 of [5] that $P(K)$ is homeomorphic to K. This would contradict that $P(g_0) \in F$ and establish the theorem.

Hence let m and ϵ be given. Let K' be the union of all elements of G contained in K and of diameter greater than or equal to ϵ. There is a positive integer p such that the components of M_p' which intersect K' are contained in K_m. Denote the union of these components by S. Since $P'(S) \subset Q'$, it follows from the proof of Theorem 2 of [2] that there is a homeomorphism h of S^3 onto itself such that: (1) if $x \in S^3 - S$, then $h(x) = x$; and (2) if $g' \in G'$ and $g' \subset S$, then diam $h(g') < \epsilon$. This homeomorphism satisfies the desired properties given in the previous paragraph and completes the proof.

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