A PROBLEM ON PARTITIONS CONNECTED WITH WARING'S PROBLEM

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1. Introduction. Let \( k, s \) be fixed positive integers, and \( n \) an arbitrary positive integer. Then we denote by \( R(n) \) the number of representations of \( n \) as a sum of \( s \) \( k \)th powers of positive integers; that is, \( R(n) \) is the number of solutions \((x_1, x_2, \cdots, x_s)\) of the Diophantine equation

\[
(1) \quad n = x_1^k + x_2^k + \cdots + x_s^k \quad (x_i \text{ positive integers}),
\]

solutions differing only in the order of the \( x_i \) being counted as distinct.

Hardy and Littlewood discovered the famous asymptotic formula

\[
(2) \quad R(n) = \frac{\Gamma^s(1 + 1/k)}{\Gamma(s/k)} \Theta(n)n^{s/k-1} + o(n^{s/k-1}) \quad (n \to \infty),
\]

where \( \Theta(n) \) is the 'singular series', and Hua [3] proved that (2) holds for \( s \geq 2k+1 \). An elegant and short proof of Hua's theorem was published, in 1948, by Estermann [2]. A more powerful method, however, was developed by Vinogradov, who showed that (2) holds for \( s \geq \lfloor 10k^2 \log k \rfloor \) provided \( k \geq 12 \) (see [7, Chapter VII]).

We have reckoned the number \( R(n) \) considering the order of the \( x_i \). If, however, we count the number of solutions of (1) without regard to the order of the summands, we get a problem of partitions. This problem seems to be open except for \( k = 1 \). When \( k = 1 \), on the other hand, there is a considerable literature on the problem (see H. Ostmann [5, p. 52], G. J. Rieger [6]).

The main purpose of the present paper is to establish the following theorem.

**Theorem 1.** Let \( P(n) \) denote the number of partitions of a positive integer \( n \) into \( s \) \( k \)th powers of positive integers. Then, for \( s \geq 2k+1 \) \((k \geq 2)\) or \( s \geq \lfloor 10k^2 \log k \rfloor \) \((k \geq 12)\), we have

\[
(3) \quad P(n) = \frac{\Gamma^s(1 + 1/k)}{s! \Gamma(s/k)} \Theta(n)n^{s/k-1} + o(n^{s/k-1}) \quad (n \to \infty).
\]

Comparing (3) with (2), it is observed that the only difference of

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the main term of $P(n)$ from that of $R(n)$ is $s!$ in the denominator. It may also be noted that our conditions on $s$ for the validity of (3) are identical with those of Hua and of Vinogradov mentioned above.

2. Henceforth we assume that $k \geq 2$ and $s \geq 2$. First, we define $R_1(n)$ as the number of solutions of (1) in which $x_1, x_2, \cdots, x_s$ are distinct, and $R_2(n)$ as the number of solutions in which at least two of the $x_i$ are equal, the order of the $x_i$ being relevant in each case. Then clearly

$$(4) \quad R(n) = R_1(n) + R_2(n).$$

Secondly, we regard (1) as a partition of $n$, and, corresponding to the above, define $P_1(n)$ as the number of partitions in which $x_1, x_2, \cdots, x_s$ are distinct, and $P_2(n)$ as the number of partitions in which at least two of the $x_i$ are equal, the order of the $x_i$ being, of course, irrelevant. Then we have also

$$(5) \quad P(n) = P_1(n) + P_2(n).$$

Moreover, it easily follows that

$$(6) \quad R_1(n) = s!P_1(n),$$

$$(7) \quad P_2(n) \leq R_2(n) \leq s! P_2(n)/2!$$

Suppose now that (2) holds for some $s$ and further that

$$(8) \quad R_2(n) = o(n^{s/k-1}).$$

Then we have, by (4),

$$(9) \quad R_1(n) = \frac{\Gamma(s+1/k)}{\Gamma(s/k)} \mathcal{S}(n)n^{s/k-1} + o(n^{s/k-1}),$$

and, by (7),

$$(10) \quad P_2(n) = o(n^{s/k-1}).$$

Therefore, we infer from (5), (6), (9), and (10)

$$(3) \quad P(n) = \frac{\Gamma(s+1/k)}{s! \Gamma(s/k)} \mathcal{S}(n)n^{s/k-1} + o(n^{s/k-1}),$$

that is, (3) follows from (2) and (8). Conversely, we can show that (8) follows from (2) and (3). Indeed, we obtain, from (4), (5), (6), and (7),

$$s!P(n) - R(n) = s!P_1(n) + s!P_2(n) - R_1(n) - R_2(n)$$

$$= s!P_2(n) - R_2(n) \geq 2R_2(n) - R_2(n) = R_2(n).$$
The left-hand side of this inequality is $o(n^{s/k-1})$ by (2), (3); and hence (8) follows. Consequently we have the following lemma.

**Lemma 1.** (3) and (8) are equivalent expressions for those values of $s$ for which (2) is valid.

3. It would be difficult, however, to calculate $R_2(n)$ precisely, and so we employ the following method:

If $Q(n)$ denotes the number of solutions of (1) (considering the order of the summands) in which $x_1 = x_2$ holds, then obviously

$$Q(n) = \int_{\alpha_0}^{\alpha_0 + 1} T^{n-2}(\alpha) T_1(2\alpha) e(-n\alpha) d\alpha \quad (\alpha_0 \text{ any real number}),$$

where

$$T(\alpha) = \sum_{x=1}^{P} e(ax^2), \quad P = \lfloor n^{1/k} \rfloor,$$

$$T_1(\alpha) = \sum_{x=1}^{P_1} e(ax^2), \quad P_1 = \lfloor (n/2)^{1/k} \rfloor; \quad e(z) = e^{2\pi iz}.$$  

More generally, it will be seen easily that $Q(n)$ equals the number of solutions of (1) in which $x_i = x_j$ for any fixed numbers $i, j (i \neq j)$ holds. Since there are $s!/2!(s-2)!$ such pairs $(i, j)$ taken from 1, 2, ..., $s$, we obtain

$$Q(n) \leq R_2(n) \leq \binom{s}{2} Q(n).$$

From (12) it follows that (8) is equivalent to

$$Q(n) = o(n^{s/k-1}).$$

4. The number $Q(n)$ can be treated by analytic methods similar to those developed for Waring’s Problem. In the first place, we shall follow the pattern of Estermann’s version [2] of Hua’s paper [3]; next we adopt Vinogradov’s method to obtain a sharper result for large $k$.

Let $a, q$ be any pair of integers such that $1 \leq a \leq q$, $(a, q) = 1$. We write $I(a, q)$ for the interval $(a - \alpha_0)/q \leq \alpha \leq (a + \alpha_0)/q$ where $0 < \alpha_0 < \frac{1}{2}$. Let $\nu$ be a real number satisfying

$$1 < \nu < (2\alpha_0)^{-1}.$$  

Then it will be verified by a slight calculation that the intervals $I(a, q)$ with $q \leq \nu$ are nonoverlapping, and hence, by (11),
\[ Q(n) = \sum_{1 \leq q \leq r} \sum_{a} J(a, q) + \int_{E} T^{a-2}(\alpha) T_{1}(2\alpha)e(-n\alpha)\,d\alpha \]
\[ = Q^{*}(n) + Q^{**}(n), \]
say, where
\[ J(a, q) = \int_{I(a, q)} T^{a-2}(\alpha) T_{1}(2\alpha)e(-n\alpha)\,d\alpha, \]
and \( E \) is the set of those numbers of the interval \( \alpha_{0} \leq \alpha < 1 \) which do not belong to any \( I(a, q) \) with \( q \leq \nu \).

Assume now that we have an estimate for \( T(\alpha) \) such that
\[ (14) \quad T(\alpha) \ll P^{1-\rho} \quad (\alpha \in E, \rho = \rho(k) > 0) \]
and also that
\[ (15) \quad \int_{0}^{1} |T(\alpha)|^t\,d\alpha \ll P^{t-k+\delta} \quad (\delta = \delta(k) > 0) \]
where \( t = t(k) \) is some positive integer. Then we obtain, for \( s \geq t+1 \),
\[ Q^{**}(n) \ll P^{(s-t-1)(1-\rho)} \int_{0}^{1} |T(\alpha)|^{t-1} |T_{1}(2\alpha)| \,d\alpha. \]
Here, by Hölder's inequality (noting the periodicity of \( T_{1}(\alpha) \)),
\[ \int_{0}^{1} |T(\alpha)|^{t-1} |T_{1}(2\alpha)| \,d\alpha \leq \left( \int_{0}^{1} |T(\alpha)|^{t}\,d\alpha \right)^{1-1/t} \left( \int_{0}^{1} |T_{1}(\alpha)|^{t}\,d\alpha \right)^{1/t}, \]
and the right member is, by (15), \( \ll P^{t-k+\delta} \). Hence we get the following estimate:
\[ (16) \quad Q^{**}(n) \ll P^{t-k-\mu}, \]
where
\[ (17) \quad \mu = 1 - \delta + (s - t - 1)\rho. \]
If we can prove that \( \mu > 0 \) for \( s \geq s_{1}(k) \), it then follows from (16) that
\[ (18) \quad Q^{**}(n) = o(n^{s_{1}(k)-1}), \]
provided \( s \geq s_{1}(k) \).

Now let us first put \( \nu = n^{1/(4k)} \) and \( \alpha_{0} = \nu/n \). Then (13) is fulfilled whenever \( n \geq 3 \). (14) and (15) are also valid with \( \rho = 2^{-k-1} - \epsilon, t = 2^{k-1}, \) and \( \delta = 1 + \epsilon \) (see [2, Lemmas 7, 4(m = k - 1)]), where \( \epsilon \) is an arbi-
trarily small positive number. Hence when $s \geq s_1(k) = 2^{k-1} + 2$, we have, by (17),

$$
\mu = -\epsilon + (s - 2^{k-1} - 1)(2^{-k-1} - \epsilon) \geq 2^{-k-1} - 2\epsilon > 0,
$$

and therefore (18) holds. We now discuss $Q^*(n)$. It will be seen that a crude estimate for $Q^*(n)$ is sufficient for our purpose. Using the trivial inequalities: $|T(\alpha)| \leq P$, $|T_1(2\alpha)| \leq P_1 < P$, we find that

$$
|J(a, q)| < \int_{I(a, q)} P^{s-1} d\alpha = 2\alpha_0 q^{-1} P^{s-1} = 2^n q^{-1} P^{s-1},
$$

from which it follows that

$$
\sum_{1 \leq q \leq P} \sum_{a} J(a, q) < 2n q^{-1} P^{s-1} \sum_{1 \leq q \leq P} q^{-1} \leq 2n q^{-1} P^{s-1}
\leq 2n^{2/(4k) + (s-1)/k-1} = 2n^{s/k-1-1/(2k)}.
$$

Thus, $Q^*(n) = o(n^{s/k-1})$, and so finally $Q(n) = o(n^{s/k-1})$ provided $s \geq 2^{k-1} + 2$.

We next turn to Vinogradov’s treatment to obtain a better result for large $k$. We put $v = P^{1-1/k}$, $\alpha_0 = (2k)^{-1} P^{1-k}$. These values again satisfy (13). By virtue of Vinogradov’s results [7, Chapter VII], we see that both (14) and (15) hold with $\rho = (3k(k-1) \log(12k^2))^{-1}$, $t = 2b(m+h)$, and $\delta = \frac{1}{2} k(k+1)\sigma$, where $k \geq 12$, $b = \lceil \frac{k}{2} \rceil$, $h = k + 2$, $\sigma = (1-1/k)^m$, and $m$ is any fixed integer greater than $k$. Let us now take

$$
m = \left[ \frac{\log(0.5k(k+1))}{-\log(1 - 1/k)} + 1 \right],
$$

which ensures that $\sigma < (0.5k(k+1))^{-1}$, whence we get $\delta < 1$. If $s \geq t + 2 = 2b(m+h) + 2$, we have therefore $\mu > (s-t-1)\rho \geq \rho > 0$. Now a simple calculation shows that

$$
2b(m+h) < 5k^2 \log k + 2.5(1 - \log 2)k^2 + 11k + 3
\leq 6k^2 \log k - 2 \quad (k \geq 12).
$$

Hence if $s \geq s_1(k) = [6k^2 \log k]$, we obtain $s > 6k^2 \log k - 1 > 2b(m+h) + 1$, so that $s \geq 2b(m+h) + 2$; and thus (18) holds. There is no difficulty in dealing with $Q^*(n)$ if we utilize an analysis analogous to that given in [7, Chapter III] (cf. Davenport [1, pp. 50–51]); indeed we can deduce that

$$
Q^*(n) = O(P^{s-1-k}) = o(n^{s/k-1}).
$$
Consequently we have $Q(n) = o(n^{s/k-1})$, provided that $s \geq \lceil 6k^2 \log k \rceil$ ($k \geq 12$).

The above arguments, together with (12), yield the following theorem.

**Theorem 2.** Let $R_2(n)$ denote the number of representations of $n$ as a sum of $s$ kth powers of positive integers where not all of the summands are distinct. Then, for $s \geq 2^{k-1} + 2$ ($k \geq 2$) or $s \geq \lceil 6k^2 \log k \rceil$ ($k \geq 12$), we have

$$R_2(n) = o(n^{s/k-1}).$$

In particular, if $s \geq 2^{k} + 1$, then (8) is valid since $2^{k} + 1 > 2^{k-1} + 2$ ($k \geq 2$) and also (2) holds by Hua's theorem. A similar argument applies to the case $s \geq \lceil 10k^2 \log k \rceil$. This proves Theorem 1 on account of Lemma 1.

**Remark.** It is noteworthy that the number $2^{k-1} + 2$, appearing in Theorem 2, is comparatively small for small values of $k$ (see the table below). It is interesting to see that the values of $2^{k-1} + 2$ for $3 \leq k \leq 6$ are respectively less than the best known upper bounds for $G(k)$ (i.e. $G(3) \leq 7$, $G(4) = 16$, $G(5) \leq 23$, $G(6) \leq 36$). As regards the case $k = 2$, a slightly better result than that of Theorem 2 holds; we have, in fact,

$$R_2(n) = O(n^{(s-3)/2+\epsilon})$$

for $s \geq 3$ (cf. Landau [4, Theorem 204], Evelyn and Linfoot [8, Lemma 2.2]).

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5. Professor H. Davenport has raised (private communication) the following question:

*If $G_0(k)$ denotes the least value of $s_0$ such that the Hardy-Littlewood formula (2) holds$^2$ for $s \geq s_0$, then does the formula (3) hold as well for $s \geq G_0(k)$?*

$^2$ In order that (2) may be an asymptotic formula for $R(n)$, it should be required that $\Theta(n) \geq c(k, s) > 0$ for all sufficiently large $n$, and also we have $G_0(k) \geq G(k)$. Thus $G_0(2) = 5$, though when $k = 2$ and $3 \leq s \leq 8$, we have exact formulae for the number of solutions of (1) if we allow the $x_i$ to be zero or negative integers (see [9]). The value of $G_0(k)$ is not known for $k > 2$.
After Lemma 1 and (12), this problem amounts to determining whether \( Q(n) = o(n^{s/k-1}) \) holds for \( s \geq G_0(k) \).

The author is unable to solve this problem completely, and we shall give here a less satisfactory answer as follows:

Formula (3) is true if \( s \geq G_0(k) + 2 \).

The proof of this is easy. For we have, if \( s \geq G_0(k) + 2 \),

\[
Q(n) = \sum_{x=1}^{P_1} R(n - 2x^k, s - 2) \ll \sum_{x=1}^{P_1} (n - 2x^k)^{(s-2)/k-1+\epsilon}
\]

\[
\ll \int_0^{(n/2)^{1/k}} (n - 2x^k)^{(s-2)/k-1+\epsilon} dx \ll n^{(s-1)/k-1+\epsilon},
\]

and thus \( Q(n) = o(n^{s/k-1}) \), where \( R(n, s-2) \) denotes the number of solutions of (1) with \( s-2 \) summands in place of \( s \), and where we have used the fact that the singular series \( \Omega(n) \), appearing in (2), is subject to the estimate \( O(n^c) \) for \( s > k \). (Actually, we can prove, by using the results [4, VI, Chapter 2, §§2, 4], that \( \Omega(n) = O((\log\log n)^c) \) \( (c = c(k) > 0) \) when \( s = k + 1 \) and \( \Omega(n) = O(1) \) when \( s > k + 1 \), provided \( k \geq 3 \).

As \( Q(n) \) is the number of solutions of

\[ 2x_1^k + x_2^k + \cdots + x_s^k = n, \]

which has \( s-1 \) variables, it is known that \( Q(n) \) satisfies an analogous asymptotic formula (see [3], [1, Theorem 4]), namely

\[
Q(n) = 2^{-1/k} \frac{\Gamma^{s-1}(1 + 1/k)}{\Gamma((s - 1)/k)} \Omega_1(n)n^{(s-1)/k-1} + o(n^{(s-1)/k-1}).
\]

It seems probable that this formula is also valid for \( s-1 \geq G_0(k) \), in agreement with formula (2). If this is true, we should have

\[
Q(n) = O(n^{(s-1)/k-1+\epsilon}) = o(n^{s/k-1})
\]

for \( s \geq G_0(k) + 1 \), which extends the validity of (3) to \( s \geq G_0(k) + 1 \).

It is quite possible\(^3\) that \( Q(n) = o(n^{s/k-1}) \) holds for \( s \geq G_0(k) \) or more values of \( s \), giving thereby an affirmative answer to our question. But this conjecture seems difficult to prove unless the actual value of \( G_0(k) \) is known.

It should be referred to in this connection that Hardy and Littlewood [10, p. 4] had introduced the 'Hypothesis K' which asserts that \( R(n, k) = O(n^\epsilon) \) for every positive \( \epsilon \). Although this hypothesis has

\(^3\) See the remark at the end of §4.
proved false when $k = 3$, it is still plausible that one has at any rate $R(n, k) = o(n^{1/k})$, which is much weaker than Hypothesis $K$ and may be compared with $Q(n) = o(n^{s/(k-1)})$ where $s = k + 1$. If the estimate $Q(n) = o(n^{s/(k-1)})$ is valid when $s = k + 1$, it may be shown by an elementary argument that the same estimate holds generally for $s \geq k + 1$. We are thus led to state the following

**Conjecture.** Let $Q(n, k)$ $(k \geq 3)$ denote the number of solutions of

$$2y_1^k + y_2^k + \cdots + y_k^k = n$$

in positive integers $y_i$. Then $Q(n, k) = o(n^{1/k})$.

**Bibliography**


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