let $M-N$ be the interior of a small $n$-cell containing $p$.) Since $X$ is $(n-2)$-connected, $H_i(X) = 0$ for $0 < i \leq n-2$ and $H_0(X) \cong \mathbb{Q}$ ([5, p. 349]). In Čech homology theory on the category of compact pairs every triad is a proper triad ([3, p. 266]). Therefore, we may apply the Mayer-Vietoris sequence to the triad $(N, (E \cup X) \cap N, M-E)$ and conclude that $H_i(M-E) = 0$, for $0 < i \leq n-2$, and $H_0(M-E) \cong \mathbb{Q}$. Since $M-E$ is a proper subset of a connected $n$-manifold, it follows that $H_i(M-E) = 0$ for $i \geq n$. (We assume here that $n > 0$. If $n = 0$, the theorem is trivial.) By Alexander duality, (see [6, p. 263]) since $E$ is arc-wise connected, $H_{n-1}(M-E) = 0$. Therefore $M-E$ is acyclic and the theorem follows.

**References**


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**A $T_1$-COMPLEMENT FOR THE REALS**

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The family of all topologies definable on an arbitrary set $X$ forms a complete lattice $\Sigma$ under the partial ordering: $\tau_1 \leq \tau_2$ if and only if $\tau_1 \subseteq \tau_2$. The lattice operations $\wedge$ and $\vee$ are defined as: $\tau_1 \wedge \tau_2 = \tau_1 \cap \tau_2$ and $\tau_1 \vee \tau_2$ is the topology generated by the base $\mathfrak{B} = \{B: B = U_1 \cap U_2, U_1 \subseteq \tau_1$ and $U_2 \subseteq \tau_2\}$. The greatest element, $1$, is the discrete topology and the least element, $0$, is the trivial topology. The lattice $\Sigma$ has been recently studied [2], [3], [4] and has been shown to be complemented [4].

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The family of all $T_1$-topologies definable on $X$ forms a complete sublattice $\Lambda$ of $\Sigma$, with greatest element $1$, and least element, the cofinite topology $\lambda = \{ U: U = \emptyset \text{ or } X - U \text{ is finite}\}$. However, an example has been given in [5] to show that $\Lambda$ is not a complemented lattice, unless $X$ is a finite set.

The question as to which $T_1$-topologies have $T_1$-complements has been studied in [1], [5], [6]. Although large classes of $T_1$-topologies have been shown to have $T_1$-complements, the most common spaces are not included. In fact, the question concerning the real numbers has been outstanding for some time.

We have shown in [6] that the space of real numbers with the usual topology has a $T_1$-complement if any countable dense subspace has a $T_1$-complement. The purpose of this paper is to use this fact to produce a complement for the reals.

Let $(R, \mu)$ be the real numbers $R$ with the usual topology $\mu$, $Q$ be the rational numbers and $D$ be the dyadic rationals. We now will define a countable dense subspace $X$ of $R$.

For each integer $k$, let $S_{0,k}$ be a sequence in $(R - D) \cap \{ (k, k + 1/2) \}$ which converges to $k$ and let $A_0 = \bigcup \{ S_{0,k} \mid k = 0, \pm 1, \pm 2, \ldots \}$.

For each integer $k$, let $S_{1,k}$ be a sequence in $(R - D) \cap \{ (k - 1/2, k) \}$ which converges to $k - 1/2$ and let $A_1 = \bigcup \{ S_{1,k} \mid k = 0, \pm 1, \pm 2, \ldots \}$.

Since $D$ is countable, $D - \{ r \mid r = k \text{ or } k + 1/2, k \text{ an integer} \}$ can be ordered as $\{ d_1, d_2, \ldots \}$. There is a bounded open interval $I_1$ containing $d_1$ such that $I_1 \cap (A_0 \cup A_1) = \emptyset$. Let $S_1$ be a sequence in $I_1 \cap (R - D)$ converging to $d_1$. Suppose $S_p$ has been chosen for each $p < n$. There is a bounded open interval $I_n$ containing $d_n$ such that $I_n \cap ((A_0 \cup A_1) \cup (\bigcup \{ S_p \mid p < n \}) = \emptyset$. Let $S_n$ be a sequence in $I_n \cap (R - D)$ converging to $d_n$.

Let $X = D \cup A_0 \cup A_1 \cup (\bigcup \{ S_n \mid n = 1, 2, \ldots \})$ and $\tau$ be the relative topology on $X$ with respect to $\mu$.

Define a topology $\tau'$ on $X$ to be the topology generated by sets of the form:

(i) $\{ x \}, x \in D$,
(ii) $U, U \in \lambda$ where $\lambda$ is the cofinite topology on $X$,
(iii) $B_i, i = 0, 1$,
(iv) $C_i, i = 1, 2, \ldots$,

where

$B_0 = A_0 \cup \{ (X - A_1) \cap (\bigcup \{ [k, k + 1/2] \mid k \text{ an integer} \}) \}$,

$B_1 = A_1 \cup \{ (X - A_0) \cap (\bigcup \{ [k - 1/2, k] \mid k \text{ an integer} \}) \}$,

and

$C_i = S_i \cup \{ (X - I_i) \cap D \}$. 

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We notice that $B_0 \cup B_1 = X$ and that $C_i \cap C_j \subseteq D$ if $i \neq j$. Since $\lambda \subseteq \tau'$, $\tau'$ is a $T_1$-topology. We will show that $\tau \vee \tau' = 1$ and $\tau \wedge \tau' = \lambda$.

(a) $\tau \vee \tau' = 1$. Let $x \in X$. If $x \in D$ then $\{x\} \in \tau$. If $x \in S_i$ then there is an open interval $U$ containing $x$ such that $U \subseteq I_i$ and $U \cap S_i = \{x\}$. Thus $\{x\} = (U \cap X) \cap C_i \in \tau \vee \tau'$. If $x \in A_0$ then there is an open interval $U$ containing $x$ such that $U \subseteq (k, k + 1/2)$ for some $k$ and $U \cap A_0 = \{x\}$. Thus $\{x\} = (U \cap X) \cap B_0 \in \tau \vee \tau'$. A similar argument holds if $x \in A_1$.

(b) $\tau \wedge \tau' = \lambda$. Let $U \subseteq \tau \wedge \tau'$ and suppose $U \neq \emptyset$. Since $U \subseteq \tau$, $U$ must contain elements of $D$; so let $x \in D \cap U$. If $x = d_n$ for some $n$, then all but a finite number of elements of $S_n$ must be in $U$. Thus almost all of $C_n \cap B_0$ or almost all of $C_n \cap B_1$ is contained in $U$ (since these are the only base elements in $\tau'$, other than $C_n$ or members of $\lambda$, which contain almost all of $S_n$). But if a cofinite subset of $C_n \cap B_1$ is in $U$ then $U$ contains almost all the integers and hence $U$ must contain a cofinite subset of $B_0$ since $B_0$ is the only base element in $\tau'$ containing the sequences which converge to the integers. But $B_0$ contains all dyadic rationals of the form $(2k + 1)/2$; so a cofinite subset of $B_1$ must be contained in $U$ and therefore $U \subseteq \lambda$.

The other cases where a cofinite subset of $C_n \cap B_0$ is contained in $U$ or where $x \in D - \{d_1, d_2, \ldots\}$ are treated in a similar fashion.

Thus $\tau'$ is a $T_1$-complement for $\tau$ and since $X$ is a countable dense subset of $R$, $\mu$ also has a $T_1$-complement. The elements of this complement may be obtained from those in $\tau'$ by following the construction given in [6].

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