A SUM INVOLVING THE MÖBIUS FUNCTION

HANSRAJ GUPTA

1. In what follows, small letters other than x denote positive integers, unless stated otherwise; x is any real positive number; p's are primes; and μ is the Mobius function.

The object of this note is to evaluate the sum

$$S_m(x) = \sum_{1 \leq n \leq x;\ (n,m)=1} \mu(n) \left[\frac{x}{n}\right]. \tag{1}$$

Without loss of generality, m and n can be taken to be square-free, and this we shall do in all that follows.

It is well known that

$$\begin{aligned} S_1(x) &= 1 \quad \text{for all } x \geqq 1, \\ S_m(x) &= 0 \quad \text{for } 0 \leqq x < 1;\ m \geqq 3. \end{aligned} \tag{2}$$

Also

$$S_m(1) = 1 \quad \text{for every } m. \tag{3}$$

We shall, therefore, consider only the case when $m > 1$ and $x \geqq 1$.

2. We have for $(m, p) = 1$,

$$\begin{aligned} S_{pm}(x) &= \sum_{1 \leq n \leq x;\ (n,m)=1} \mu(n) \left[\frac{x}{n}\right] - \sum_{1 \leq n \leq x/p;\ (n,pm)=1} \mu(pn) \left[\frac{x}{pn}\right], \\ &= \sum_{1 \leq n \leq x;\ (n,m)=1} \mu(n) \left[\frac{x}{n}\right] + \sum_{1 \leq n \leq x/p;\ (n,pm)=1} \mu(n) \left[\frac{x/p}{n}\right], \end{aligned}$$

$$= S_m(x) + S_{pm}(x/p), \tag{4}$$

$$= \sum_{i=0}^{c} S_m\left(\frac{x}{p^i}\right), \quad \text{where} \quad c = [\log_p x]. \tag{5}$$

In particular, taking $m = 1$ in (5),

$$S_p(x) = \sum_{i=0}^{c} S_1\left(\frac{x}{p^i}\right) = 1 + c = [\log_p (px)]. \tag{6}$$

(This result is due to M. Newman.) Again,

Received by the editors December 18, 1966.

$$S_{p_1p_2}(x) = S_{p_1}(x) + S_{p_1p_2}(x/p_2),$$
$$= S_1(x) + S_{p_1}(x/p_1) + S_{p_2}(x/p_2) + S_{p_1p_2}(x/p_1p_2).$$

In general,

$$S_m(x) = \sum_{d|m} S_d\left(\frac{x}{d}\right); \tag{7}$$

and, if $(m_1, m_2) = 1$,

$$S_{m_1m_2}(x) = \sum_{d|m_2} S_{m_1d}\left(\frac{x}{d}\right). \tag{8}$$

3. It might be observed that (5) enables us to reduce $S_m(x)$ step by step to a sum of terms of the form $S_1(x/t)$ where t runs through all those numbers $\leqq x$, which are of the form $p_1^{x_1}p_2^{x_2} \cdots p_k^{x_k}$, where x_i's are integers $\geqq 0$, p's being all the distinct prime divisors of m. Since $S_1(x/t) = 1$ for each such t, we have the

THEOREM. *$S_m(x)$ is the number of those divisors of m^h, $h = [\log_2 x]$, which do not exceed x.*

As an alternate proof of this theorem, we offer the following. For $j \geqq 2$, we have

$$S_m(j) - S_m(j-1) = \sum_{1 \leq n \leq j;\ (n,m)=1} \mu(n) \left\{\left[\frac{j}{n}\right] - \left[\frac{j-1}{n}\right]\right\}. \tag{9}$$

Now $[j/n] - [(j-1)/n] = 1$ or 0, according as n does or does not divide j. First, let j be a divisor of m^h. Then every term in the sigma on the right of (9) is zero except the one with $n = 1$. Hence

$$S_m(j) = 1 + S_m(j-1). \tag{10}$$

Next, let j be not a divisor of m^h. If in the canonical factorization of j, exactly q ($\geqq 0$) primes are different from those that occur in m, then in (9) $[j/n] - [(j-1)/n] = 1$ whenever n is a square-free product of some or all of the q primes or when $n = 1$, otherwise $[j/n] - [(j-1)/n] = 0$. Hence, in this case,

$$\begin{aligned} S_m(j) - S_m(j-1) &= 1 - \binom{q}{1} + \binom{q}{2} - \cdots + (-1)^q\binom{q}{q}, \\ &= 0. \end{aligned} \tag{11}$$

From (10) and (11), we have

$$S_m(j) = \sum_{1 \leq t \leq j;\, t | m^h} 1.$$

This proves the theorem.

4. **An asymptotic formula for $S_m(x)$.** Let x be sufficiently large, and $p_1 < p_2 < p_3 < \cdots < p_k$ be all the prime divisors of m. Then $p_1^{x_1} p_2^{x_2} \cdots p_k^{x_k} \leqq x$ if and only if

$$x_1 \log p_1 + x_2 \log p_2 + \cdots + x_k \log p_k \leqq \log x. \tag{12}$$

Hence $S_m(x)$ is the number of solutions in nonnegative integers, of the inequality (12).

This is the same as the number of lattice points on and within the k-dimensional polyhedron formed by the hyperplane

$$\sum_{i=1}^{k} x_i/\log_{p_i} \qquad x = 1,$$

and the axes of coordinates. The number of lattice points being asymptotically equal to the volume of the polyhedron, we have

$$S_m(x) \sim \frac{1}{k!} \prod_{i=1}^{k} \alpha_i, \quad \text{where } \alpha_i = i + \log_{p_i} x. \tag{13}$$

In fact

$$S_m(x) = \frac{1}{k!} \prod_{i=1}^{k} \alpha_i + O\left(\prod_{i=1}^{k-1} \alpha_i \right), \tag{14}$$

the error term being of the order of the number of lattice points on the boundary.

University of Arizona and
Allahabad University, India