A GENERALIZATION OF A THEOREM OF HANS LEWY
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Let \( p \) be a real-valued \( C^\infty \) function defined in a neighborhood of the origin \( 0 \) in \( \mathbb{C}^n \), such that \( p(0) = 0 \), \( d\rho(0) \neq 0 \). Then, near zero, \( M = \{ z; \rho(z) = 0 \} \) is a real submanifold of \( \mathbb{C}^n \) of dimension \( 2n - 1 \). If \( \partial \overline{\partial} \rho(0) \neq 0 \), then \( M \) has a holomorphic hull which contains an open set. We shall prove an \( L^2 \) version of this fact. Let \( \overline{\partial} \) represent the tangential Cauchy-Riemann operator on \( M \) introduced by Kohn [1]. By \( L^2 \) on \( M \), we mean the space of functions which are square integrable with respect to surface area.

**Theorem.** There is a neighborhood \( N \) of \( 0 \) such that if \( D = \{ z \in N; \rho(z) < 0 \} \) and \( f \) is an \( L^2 \) function on \( N \cap M \), the following are equivalent:

(i) \( f \) is a weak solution of the equation \( \overline{\partial} f = 0 \),

(ii) \( \int_M f \overline{\partial} \alpha = 0 \) for all \( (n, n-2) \) forms \( \alpha \) whose support intersects \( N \cap M \) in a compact set,

(iii) \( f \) is the boundary value of a function holomorphic in \( D \),

(iv) \( f \) is locally \( L^2 \) approximable by functions holomorphic in a neighborhood of \( N \cap M \).

The only nontrivial part of this theorem is (ii) implies (iii) and (iv); this was proven in [2] by Hans Lewy for \( f \) a \( C^1 \) function \( (n = 2, \text{but that does not matter}) \). The proof here is an adaptation of his argument. We need to use the following verifiable lemmas.

**Lemma 1.** Let \( f \) be a square integrable function defined in a domain in \( \mathbb{C}^n \). \( f \) is holomorphic if and only if \( \int f \overline{\partial} \alpha = 0 \) for all compactly supported \( (n, n-1) \) forms \( \alpha \).

**Lemma 2.** Let \( X \) be a compact Hausdorff space, \( \mu \) a finite Baire measure, \( \Delta = \{ z \in C; |z| < 1 \} \), \( \Gamma = \{ z \in C; |z| = 1 \} \). Let \( f: X \to L^2(\Gamma) \) be square integrable; \( \int \| f(x) \|^2 d\mu < \infty \) and suppose also

\[
\int_{\Gamma} f(x)(\theta)e^{in\theta} d\theta = 0 \quad \text{for } n > 0.
\]

Let \( \hat{f}(x, z) \) for \( z \in \Delta \) be the Cauchy integral of \( f(x) \). Then \( \hat{f} \) has the bound-

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ary values \( f \). More precisely, let \( \phi: X \times \Gamma \times [0, 1] \to \Delta \) be a continuous map with these properties:

(i) \( \phi(x, \theta, t) = e^{it\theta}, \phi(x, 0) = 1, \)

(ii) \( \phi(X \times \Gamma \times (0, 1)) \subset \Delta. \)

Then

\[
\lim_{\delta \to 0} \int_{X \times \Gamma} |\phi(x, \phi(x, \theta, \delta)) - f(x, \theta)|^2 d\delta d\mu = 0.
\]

Now, we return to \( M \). Because \( \partial \bar{\partial} \rho(0) \neq 0 \) and \( d\rho(0) \neq 0 \) we may choose complex coordinates \((z, w, w_1, \cdots, w_{n-2})\) near zero so that \( M \) is given by

\[
0 = \rho(z) = \text{Re } w + z\bar{z} + Q(w, z) + O(2),
\]

where \( Q \) is the multivariable \((w_1, \cdots, w_{n-2})\), \( Q \) is a quadratic form, and \( O(2) \) consists of terms of higher order at 0. Let \( \pi: C^n \to C^{n-1}, \pi(z, w, u) = (w, z) \). It is shown in [3], that the mapping \( \pi \) has the following structure. There is a ball \( N \), center at 0 such that \( \pi(M \cap N) \) is the closure (in \( \pi(N) \)) of a domain \( D_0 \). For \((w, u) \in D_0, \Gamma_{(w, u)} = \pi^{-1}(w, z) \cap M \) is a simple closed curve in the \( z \)-plane bounding the domain \( \Delta_{(w, u)} \). As \((w, u) \to bD_0, \Gamma_{(w, u)} \to \)point. Let \( D = \{(z, w, u); (w, u) \in D_0, z \in \Delta_{(w, u)} \}\). With the situation so given we prove

**Lemma 3.** Let \( f \in L^2 \) on \( M \) with the property (ii) of the theorem. Then

\[
j \in L^2 D \text{ is holomorphic in } D. \text{ If } B \text{ is a closed ball contained in } N, j \in L^2 B \text{ is } L^2-( \text{approximately by translates of } j \text{ which are holomorphic in a neighborhood of } B \cap M). \]

**Proof.** First of all, \( j \) is clearly locally \( L^2 \) in \( D \). We use Lemma 1 to verify that \( j \) is holomorphic. Let \( \beta \) be a compactly supported (in \( D \)), \((n, n-1)\) form. Let \( dV' = dw \wedge dw_1 \wedge \cdots \wedge dw_n \wedge dw \wedge \cdots \wedge dw_n \).

(i) \( \beta = h dz \wedge dV' \). Then

\[
\int j \bar{\beta} = \int_{D_0} \left\{ \int_{\Delta_{(w, u)}} \left[ \frac{1}{2\pi i} \int_{\Gamma_{(w, u)}} \frac{\partial h}{\partial \bar{z}} f(\eta, w, z) d\eta \right] \bar{z} \wedge dz \right\} dV'.
\]

Since \( \eta \) is outside the support of \( h \), by Lemma 1 for \( n = 1 \), the innermost integral is always zero. Thus \( j \bar{\beta} = 0 \) in this case.

(ii) \( \beta = d\bar{z} \wedge dz \wedge \gamma \), where \( \gamma \) is compactly supported \((n-1, n-2)\)
with no \(dz\) or \(d\bar{z}\) term. Then \(\overline{\partial} \beta = d\bar{z} \wedge dz \wedge \overline{\partial}' \gamma\) where \(\overline{\partial}' \gamma\) is taken as if \(\gamma\) were considered as an \((n-1, n-2)\) form in the \((w, \xi)\)-space, with coefficients varying in \(z\).

Thus

\[
\int f \overline{\partial} \beta = \pm \frac{1}{2\pi i} \int_{D_0} \left\{ \int_{\Delta_{(w,\xi)}} \left[ \int_{\Gamma_{(w,\xi)}} \frac{f(\eta, w, \xi) d\eta}{\eta - z} \wedge \overline{\partial} \gamma \right] d\bar{z} \wedge dz \right\}
\]

\[
= \pm \frac{1}{2\pi i} \int_{D_0} \left\{ \int_{\Gamma_{(w,\xi)}} f(\eta, w, \xi) \left[ \int_{\Delta_{(w,\xi)}} \frac{\overline{\partial}' \gamma}{\eta - z} d\bar{z} \wedge dz \right] d\eta \right\}.
\]

Let

\[
\alpha(\eta, w, \xi) = \left( \int_{\Delta_{(w,\xi)}} \frac{\gamma}{\eta - z} d\bar{z} \wedge dz \right) \wedge d\eta.
\]

\(\alpha\) is a \(C^\infty(n, n-2)\) form defined in a neighborhood of \(M\) whose support intersects \(M\) in a compact set (since \(\gamma\) vanishes in a neighborhood of \(M\)). Further, computing \(d\alpha\), we find \(\int f \overline{\partial} \alpha = \int_M f \overline{\partial} \alpha = 0\) by hypothesis.

Now, since any compactly supported \((n, n-1)\) form on \(D\) is a sum of forms of type (i) and (ii), we have \(\int f \overline{\partial} \beta = 0\) for all such forms, so by Lemma 1, \(f\) is holomorphic.

Now, in order to apply Lemma 2, we must verify that, for fixed \((w, \xi)\), \(f(z, w, \xi)\) has the boundary value \(f\). That is

\[(*) \quad \int_{\Gamma_{(w,\xi)}} f(z, w, \xi) dz = 0 \quad \text{for } n \geq 0.\]

Let

\[
F(w, \xi) = \int_{\Gamma_{(w,\xi)}} f(z, w, \xi) dz \quad w \in D_0,
\]

\[
= 0 \quad w \in D_0.
\]

We show that \(F\) is holomorphic in \(\pi(N)\). First, if \(\beta\) is a \(C^\infty(n-1, n-2)\) form, compactly supported in \(D_0\),

\[
\int F \overline{\partial} \beta = \int_{D_0} \int_{\Gamma_{(w,\xi)}} f(z, w) dz \wedge \overline{\partial} \beta = \int_M f \overline{\partial} (dz \wedge \beta) = 0.
\]

Let \(\beta\) now be any \((n-1, n-2)\) form compactly supported in \(\pi(N)\). Choose real \(C^\infty\) coordinates in \(\pi(N)\), \(x_1, \ldots, x_{2n-2}\) so that \(bD_0 = \{x_1 = 0\}, \ D_0 = \{x_1 > 0\}\). (We may have to do this locally, but after applying a partition of unity to \(\beta\) this is the general case.) Reducing to the plane \(x_2, \ldots, x_{2n-2} = \text{constant}\), \(\arg z_1 = \text{constant}\), we see that \(M\) intersects this plane in a curve \(x_1 = A |z_1|^2 + \cdots\), where \(A\) depends
differentiably on the other constants and is bounded away from zero. Thus the length of \( \Gamma_\omega \) is of the order of \( 2\pi \sqrt{x_1} \).

Now let \( \rho(x_1) \) be a \( C^\infty \) function such that

\[
\rho = 0 \quad \text{when} \quad x_1 \leq \epsilon, \quad \left| \frac{d\rho}{dx_1} \right| \leq 2/\epsilon.
\]

Now \( \int_{D_0} F \delta \beta = \int_{D_0} F \delta (\rho \beta) \), since \( \int F \delta (1-\rho) \beta = 0 \), as above. Now \( \delta \rho \beta = \delta \rho \beta + \rho \delta \beta \),

\[
\left| \int F \delta \rho \wedge \beta \right| = \left| \int_{D_0} \int_{\Gamma_{(w,\xi)}} f(z, w, \xi) dz \wedge \delta \rho \wedge \beta \right|
\]

\[
= \left| \int_{D_0} \int_{\Gamma_{(w,\xi)}} \left\{ f(z, x) x^n \frac{d\rho}{dx_1} \xi_\beta(x) \right\} dz \wedge dV \right|
\]

where \( \xi_\beta \) is a function depending only on \( x \), and \( dV \) is the element of volume in \( D_0 \). Using Schwarz's inequality,

\[
\left| \int F \delta \rho \wedge \beta \right| \leq K_n \| f \| \left( \int_{D_0} \left| \frac{\partial \rho}{\partial x_1} \xi_\beta(x) \right|^2 \int_{\Gamma_{(w,\xi)}} dV \right)^{1/2}.
\]

But \( \int_{D_0} \delta |z| \sim 2\pi \sqrt{x_1} \), thus

\[
\left| \int F \delta \rho \wedge \beta \right| \leq K_n' K_\beta \| f \| \epsilon^{-1} \int_{0 \leq x_1, x_1 \leq \epsilon} x_1 dx_1 \leq \| f \| \sqrt{\epsilon}.
\]

Now \( \int F \delta \beta \) is even better, so we find, letting \( \epsilon \to 0 \) that \( \int_{D_0} F \delta \beta = 0 \). Thus \( F \) is holomorphic in \( \pi(N) \), and since it is identically zero in an open set, it is identically zero, and (*) is verified.

Now, fix a Riemann map \( R_{(w,\xi)} \) of \( \Delta_{(w,\xi)} \) onto \( \{|z|<1\} \), differentiable at the boundary, and varying differentiably in \( (w, \xi) \). Define \( \psi : \overline{D} \to M \), \( \psi(z, w, \xi) \) = the point on \( \Gamma_{(w,\xi)} \) with the same Riemann mapping argument as \( (z, w, \xi) \).

Now, it is easy to verify that for \( \delta > 0 \), if \( (z, w, \xi) \in M \), \( (z, w-\delta, \xi) \in D \). Let \( \phi(w, \xi, \theta, \delta) \) be the point in \( \Gamma_{(w-\delta,\xi)} \cap (M-\delta) \) whose Riemann mapping argument is \( \theta \). By the lemma,

\[
\lim_{\delta \to 0} \int_{D_0 \times \Gamma} |\dot{f}(\phi(w, \xi, \theta, \delta)) - f(R_{(w-\delta,\xi)}(e^{i\theta}), w - \delta, \xi)|^2 dV = 0,
\]

or what is the same

\[
\lim_{\delta \to 0} \int_M |\dot{f}(z, w - \delta, \xi) - f(\psi(z, w - \delta, \xi))|^2 dV = 0.
\]
Now the mapping \((z, w, \xi) \rightarrow \psi(z, w - \delta, \xi)\) is a differentiable family of transformations on \(M\), tending to the identity as \(\delta \to 0\). Thus
\[
\lim_{\delta \to 0} \int_M |f(\psi(z, w - \delta, \xi)) - f(z, w, \xi)|^2 dV = 0.
\]
Thus \(f(z, w - \delta, \xi) \to f(z, w, \xi)\) as \(\delta \to 0\) in \(L^2\) on \(M\), and since \(M - \delta \subset D\), \(f(z, w - \delta, \xi)\) is holomorphic on \(M\).

**Bibliography**


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