CORRECTION TO "THE EXISTENCE OF PROPER SOLUTIONS OF A SECOND ORDER ORDINARY DIFFERENTIAL EQUATION"

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I am grateful to Professor L. K. Jackson for pointing out an error in the proof of the lemma and in the proof of Theorem 2 in my paper [1]. Here I shall give a new version of the lemma and a second proof of the Theorem.

The error arises from ignoring equations with solutions which have a finite maximal interval of existence. In the lemma, such a solution could remain within the triangle \( T \) for its entire interval of existence; in Theorem 2, there is a solution that remains in the region \( T \) for its entire interval of existence, as guaranteed by Ważewski's Theorem, but that interval might be finite. Jackson provided the example

\[
x'' = x + (2 + 8(1 - t^2))(x')^2 + 2|1 - t|^{1/2}x' \quad (t \geq 0)
\]

and the solution \( x(t) = 1 + (1-t)^{1/2} \) which has \([0, 1)\) as its maximal interval of existence. (In the lemma, if \( A = 2 \) and \( c = 4 \), then \( M = 2 \) and \( d = 8.5 \).)

**Lemma (New Version).** Given \( A > 0 \) and \( 0 \leq a < c \) there exists a \( d(A, c) > 0 \) such that if \( x(t) \) is a solution of (1) with \( 0 < x(a) < (-Aa/c) + A, x'(a) \leq -d \), then either \( x(b) = 0 \) for some \( b \in (a, c) \), or \( x(t) \) has a finite maximal interval of existence \( [a, b) \subseteq [a, c) \) and \( x(t) \to x_0 \) \( (0 \leq x_0 < (-Ab/c) + A) \) \( x'(t) \to -\infty \) as \( t \to b \).

**Proof.** Because of the assumptions on (1), a solution \( x(t) \) will have a finite maximal interval of existence only if \( x(t) \to \pm \infty \) or \( x'(t) \to \pm \infty \) at some finite \( t \).

Let \( \tau = \{(t, x): 0 \leq t \leq c, 0 \leq x \leq (-At/x) + A \} \) and let \( H \) be the hypotenuse of \( \tau \). Let \( x(t) \) be a solution of (1) with \( 0 < x(a) < (-Aa/c) + A \). If \( x(t) \) leaves \( \tau \), it either (a) crosses \( H \), or (b) crosses the \( t \)-axis, say at \( t = b \). (This covers the case of \( x(t) \to \pm \infty \) at some finite \( t \).) If \( x(t) \) remains in \( \tau \), it has a finite maximal interval of existence, say \( [a, b) \), and either (c) \( x'(t) \to +\infty \), or (d) \( x'(t) \to -\infty \) as \( t \to b \).

In a manner similar to that used in the original lemma, we find a \( d(A, c) > 0 \) such that \( x'(a) \leq -d \) implies that \( x'(t) < -A/c \) for \( a \leq t < b \). Such an \( x(t) \) can satisfy neither (a) nor (c).

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NEW PROOF OF THEOREM 2. We delete the last paragraph of the old proof and we cannot use Ważewski's Theorem. Continuing from the bottom of page 596:

Let \( W = \{ (t, x, y): t = 0, x = A, -d \leq y \leq 0 \} \) where \( d \) is determined by the lemma with \( a = 0, c > 0 \) arbitrary. Let \( X \subseteq W \) be such that if \((x(t), y(t))\) is a solution of (4) with \((0, x(0), y(0)) \in X\), then there exists a \( t_0 \), \( 0 \leq t_0 < \infty \), such that \((x(t), y(t))\) is defined for \( 0 \leq t \leq t_0 \) and \((t_0, x(t_0), y(t_0)) \in Q\); let \( Y \subseteq W \) be defined like \( X \) except that \((0, x(0), y(0)) \in Y \) implies that \((t_0, x(t_0), y(t_0)) \in R\); and let \( Z \subseteq W \) be defined like \( X \) except that \((0, x(0), y(0)) \in Z \) implies that \((x(t), y(t))\) is defined for \( 0 \leq t < t_0 \) and \( x(t) \to x_0 \neq 0, y(t) \to -\infty \) as \( t \to t_0 \).

Solutions of (4) with \((0, x(0), y(0)) \in W\) may leave \( T \) (the \( X \) and \( Y \) initial values—included here are solutions with finite maximal intervals of existence and for which \( x(t) \to \pm \infty \), or \( y(t) \to \infty \) at some finite \( t \); they may remain in \( T \) and have a finite maximal interval of existence (the \( Z \) initial values); or they may remain in \( T \) and be defined for \( 0 \leq t < \infty \) (the proper solutions of (1)).

Now \( X \) is nonempty since \((0, A, 0) \in X\), \((Y \cup Z)\) is nonempty by the lemma, and \( X \) and \((Y \cup Z)\) are disjoint. We shall show that \( X \) and \((Y \cup Z)\) are open relative to \( W \). It then follows that \( W \neq X \cup (Y \cup Z) \) and hence (1) has a proper solution.

Assumption (i) for (1) implies that the solutions of (4) depend continuously upon initial conditions. Let \((0, x_0, y_0) \in X\) and let \((x(t), y(t))\) be the solution of (4) with \((0, x(0), y(0)) = (x_0, y_0)\). There exists a \( t_1 \) such that \((x(t), y(t))\) is defined for \( 0 \leq t \leq t_1 \) and \((t_1, x(t_1), y(t_1))\) is an element of the complement of the closure of \( T \). Let \((0, u_0, v_0) \in W\) and let \((u(t), v(t))\) be the solution of (4) with \((0, u(0), v(0)) = (u_0, v_0)\). We can choose \( \delta > 0 \) such that \(|x_0-u_0| + |y_0-v_0| < \delta \) implies that \((u(t), v(t))\) is defined on \[0, t_1\], \((t_1, u(t_1), v(t_1))\) is an element of the complement of the closure of \( T \), and the point where \((t, u(t), v(t))\) egresses from \( T \) lies in \( Q \). Therefore \( X \) is open relative to \( W \).

Likewise \( Y \) is open relative to \( W \).

Let \((0, x_0, y_0) \in Z\) and let \((x(t), y(t))\) be the solution of (4) with \((0, x(0), y(0)) = (x_0, y_0)\). There exists a \( t_1 \), \( 0 < t_1 < \infty \), such that \((x(t), y(t))\) exists for \( 0 \leq t < t_1 \) and \( x(t) \to x_1 \) \((0 \leq x_1 < A)\), \( y(t) \to -\infty \) as \( t \to t_1 \).

In the \( t, x \)-plane consider the open triangle \( \sigma \) with vertices \((0, 0), (0, 2A), (t_2, 0)\) where \( t_2 \) is chosen so that \((t_1, x_1) \in T\). By the lemma, choose \( d_1 = d_1(2A, t_2) > 0 \). There exists a \( t_3 \), \( 0 < t_3 < t_1 \), such that \((t_3, x(t_3)) \in \sigma \) and \( y(t_3) \leq -2d_1 \). Let \((0, u_0, v_0) \in W\) and let \((u(t), v(t))\) be the solution of (4) with \((0, u(0), v(0)) = (u_0, v_0)\). There exists a \( \delta > 0 \) such that \(|x_0-u_0| + |y_0-v_0| < \delta \) implies that \((u(t), v(t))\) is defined for \( 0 \leq t \leq t_3 \), \((t_3, u(t_3)) \in \sigma \), and \( v(t_3) \leq -d_1 \). Apply the lemma to
(u(t), v(t)): there exists a \( t_4 \in (t_3, t_2) \) such that either \( u(t_4) = 0 \), or \( u(t) \to u_1 \geq 0, v(t) \to -\infty \) as \( t \to t_4 \). In either case \((0, u_0, v_0) \in (Y \cup Z)\). And since \( Y \) is open relative to \( W \) it follows that \( Y \cup Z \) is open relative to \( W \).

Professor Jackson has also sent me a version of Theorem 2, and he permits \( f(t, x, y) \) to be either nondecreasing or nonincreasing in \( y \) for each fixed \( t, x \), which he proves using the theory of sub- and superfunctions.

Reference


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