**M-SIMILARITY AND ISOMORPHISMS IN $B_0$-SPACES**

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Let $X$ be a $B_0$-space, i.e. a locally convex Frechét space. A system $(x_i; f_i)$ with $(x_i) \subset X$ and $(f_i) \subset X^*$ is a **generalized basis** for $X$ if (a) $f_i(x_j) = \delta_{ij}$ for all $i$ and $j$, and (b) $f_i(x) = 0$ for all $i$ implies $x = 0$ [2]. The sequence space $F(X) = \{(f_i(x)) : x \in X\}$ becomes a $B_0$-space equivalent to $X$ when equipped with the topology induced by the map $F(\cdot) = (f_i(\cdot))$. Arsove and Edwards proved that a space $X$, with generalized basis $(x_i; f_i)$, is isomorphic (i.e., linearly homomorphic) to a space $Y$ if and only if $Y$ has a generalized basis $(y_i; g_i)$ such that, as sets, $F(X) = G(Y)$. The author discussed [3] a concept dual to that of generalized basis, and obtained isomorphism theorems analogous to the Arsove-Edwards theorem. W. Ruckle [6] discussed matrix maps which preserve bases in $B_0$-spaces. (A system $(x_i; f_i)$ is a basis for $X$ if it is a generalized basis, and $x = \sum f_i(x)x_i$ for every $x$ in $X$.)

Here we introduce a generalized notion of similarity and obtain results which contain the above mentioned work in the context of $B_0$-spaces.

**Definition.** Let $(x_i; f_i)$ be a biorthogonal system for a $B_0$-space $X$ (i.e., $f_i(x_j) = \delta_{ij}$ for all $i, j$), and $(y_i; g_i)$ such a system for a $B_0$-space $Y$. We shall say that $(x_i; f_i)$ and $(y_i; g_i)$ are **$M$-similar** if either (a) there exists a matrix $A: F(X) \rightarrow G(Y)$ which is one-to-one and onto, or (b) there exists a matrix $B: G(Y) \rightarrow F(X)$ which is one-to-one and onto. If $U$ is a $B_0$-space of sequences, let $U' = \{(t_i) : \sum t_iu_i$ converges for all $(u_i)$ in $U\}$.

**Theorem 1.** If the $B_0$-spaces $X$ and $Y$ have $M$-similar generalized bases, $(x_i; f_i)$ and $(y_i; g_i)$, respectively, then $X$ and $Y$ are isomorphic.

**Proof.** By hypothesis, we may assume that there is a matrix $A: F(X) \rightarrow G(Y)$ which is one-to-one and onto. Since $F(X)$ and $G(Y)$ are $FK$-spaces (see, eg. [9, p. 202]), we conclude that $A$ defines an isomorphism.

A converse statement is provided by the Arsove-Edwards theorem, since similar generalized bases are trivially $M$-similar. The following is a somewhat stronger result.

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**Theorem 2.** Let $X$ and $Y$ be $B_0$-spaces isomorphic under $T: X \to Y$. If $(x_i; f_i)$ and $(y_i; g_i)$ are generalized bases for $X$ and $Y$, respectively, they are $M$-similar if either (a) $((F^{-1})^*T^*)(g_i) \in F(X)'$ for each $i$, or (b) $((G^{-1})^*(T^{-1})^*)(f_i) \in G(Y)'$ for each $i$.

**Proof.** For any sequence $\alpha$ in $F(X)$, we have $(GTF^{-1})(\alpha) = \beta$ in $G(Y)$ with $\beta_i = g_i(TF^{-1})(\alpha)$. Assuming (a), $((F^{-1})^*T^*)(g_i) = (\gamma_{ij}) \in F(X)'$, and $\beta_i = \sum_{k=1}^n \gamma_{jk} \alpha_k$. The matrix $\Gamma = (\gamma_{ij})$ therefore gives the isomorphism $GTF^{-1}$.

Ruckle [7] showed that if $(x_i; f_i)$ is a basis for $X$, then $F(X)' = (F(X))^*$ with the natural representation. From this, we get

**Corollary 1.** Let $X$, $Y$, $(x_i; f_i)$ and $(y_i; g_i)$ be as in Theorem 2. If $(x_i; f_i)$ is a basis for $X$, then $(x_i; f_i)$ and $(y_i; g_i)$ are $M$-similar.

Another immediate consequence of the last theorem is the following:

**Corollary 2.** Let $U$ and $V$ be isomorphic FK-spaces. The isomorphism $T: U \to V$ has a matrix representation if $T^*(\delta^i) \in U'$ for all $i$, where $\delta^i(v) = v_i$ for all $i, v \in V$.

In the special case that $(x_i; f_i)$ is a basis for $X$, $M$-similar to a generalized basis $(y_i; g_i)$ for $Y$, it is easy to get the analog of Ruckle’s theorem [6, p. 548]. Let $S^0_Y = \{ (t_i): \sum_{j=1}^n t_{ij} y_j \text{ converges in } Y \}$. Using Theorem 1 and Ruckle’s proof, we get the following

**Proposition 1.** If $(x_i; f_i)$ is a basis for the $B_0$-space $X$, and if $(y_i; g_i)$ is a generalized basis for the $B_0$-space $Y$, which is $M$-similar to $(x_i; f_i)$, then $(y_i; g_i)$ is a basis for $Y$ if and only if $A\alpha \in S^0_Y$ for every $\alpha \in F(X)$, where $A$ is the matrix guaranteed by $M$-similarity and Corollary 1.

It is worth noting that $M$-similarity does not preserve fundamental sequence: All $B_0$-spaces have nontotal generalized bases (see, e.g., [3]), and if $(x_i; f_i)$ is a basis for $X$, $(y_i; g_i)$ a nontotal generalized basis for $Y$, Corollary 1 tells us that the systems are $M$-similar. Let $(x_i; f_i)$ be a biorthogonal system for $X$, and let $\hat{x}_i$ be the canonical image of $x_i$ in $X**$. We call $(x_i; f_i)$ a dual generalized basis for $X$ if $(f_i; \hat{x}_i)$ is a generalized basis for $X^*$ [3]. We shall call biorthogonal systems $(x_i; f_i)$ and $(y_i; g_i)$ $M^*$-similar if $(f_i; \hat{x}_i)$ and $(g_i; f_i)$ are $M$-similar. With this definition, Theorem 1, and the proof of Theorem 2 in [3] we get

**Theorem 3.** Let $(x_i; f_i)$ and $(y_i; g_i)$ be $M^*$-similar dual generalized bases for the $B_0$-spaces $X$ and $Y$, respectively. Then $X$ and $Y$ are isomorphic.
It is clear that the analog to Theorem 2, above, holds for $M^*$-similarity.

A biorthogonal system which is both a generalized basis and a dual generalized basis is called a Markuševič basis. A Markuševič basis exists in any separable linear topological space with a total sequence of continuous linear functionals [4], [8]. As noted above, a Markuševič basis can be $M$-similar to a nontotal generalized basis. Following [3], however, we easily obtain the following:

**Proposition 2.** If $(x_i; f_i)$ is a Markuševič basis for $X$ which is simultaneously $M$-similar and $M^*$-similar to the generalized basis $(y_i; g_i)$ (for $Y$), then $(y_i; g_i)$ is a Markuševič basis for $Y$.

There is a partial converse to this statement. If $A: U \rightarrow V$ is one to one and onto, where $U$ and $V$ are FK-spaces, then the rows of $A$ are in $U'$ and the columns of $A$ are in $V$. With this fact, and the proof of Theorem 2, the following proposition is immediate.

**Proposition 3.** Let $(x_i; f_i)$ and $(y_i; g_i)$ be $M$-similar Markuševič bases (with matrix $A: F(X) \rightarrow G(Y)$). The systems are $M^*$-similar if $G(Y) \subseteq \bar{Y} (Y^*)'$, where $\bar{Y} (Y^*) = \{ (g(y_i)): g \in Y^* \}$.

It is easy to see that $M$-similar systems are similar if the matrix involved is the identity matrix. Paley-Wiener theorems (see, e.g., [1]) suggest that $M$-similar systems are similar for a large class of matrices. The author knows no nontrivial characterization of this class.

Other problems arise in connection with $M$-similarity since the relation is not transitive. To see this, let $A: U \rightarrow V$, with $U$ and $V$ FK-spaces, be a matrix isomorphism with no inverse; that is, there is no matrix $B: V \rightarrow U$ with $B(Au) = u$ for all $u \in U$ (see [9, p. 228]). If $C: W \rightarrow V$ has the same property, the sequences $(\delta_{ij})_{i=1}^n = e_i$ are $M$-similar in $U$ and $V$ and in $V$ and $W$, but not necessarily in $U$ and $W$.

**References**


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