INDUCED MONOMIAL REPRESENTATIONS, YOUNG ELEMENTS, AND METACYCLIC GROUPS

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1. **Introduction.** In [1, §28] Curtis and Reiner, after having constructed a full set of pairwise nonequivalent, irreducible left ideals of the group-algebras QS_n by means of the so-called Young diagrams, raise the question whether the same set of ideas can be applied to other classes of finite groups to give every (isomorphism class of) irreducible representation. In this paper we give an affirmative answer to the question in the sense that we show that the idea can give the representations of the metacyclic groups. The outline of the paper is as follows: In §2 we associate with each quadruple (P, R, π, ρ), consisting of a pair of subgroups P and R of a finite group G, and linear characters $\pi: P \to K$, $\rho: R \to K$, and satisfying certain compatibility conditions (see (2.1–2)) an element e of the group algebra KG, such that the left ideal $(KG)e$, generated by e, is minimal. Imposing further conditions on P and R we investigate in §3 the question of equivalence between left ideals of the form $(KG)e$. In §4 we give an explicit basis of such ideals, and we identify the corresponding representations as "induced monomial representations." Finally in §5 we prove that in case of a metacyclic group G the method gives a full set of nonequivalent, irreducible representations.

As for notation we mostly follow the book [1].

2. **Construction of Young elements.**[1]

THEOREM 2.1. *Let P and R be subgroups of the finite group G; let K be any field of characteristic $p \geqq 0$ with $p \nmid [G:1]$, and let $\pi: P \to K$, $\rho: R \to K$ be linear characters of P and R, respectively. Suppose that*

$$\pi \mid P \cap R = \rho \mid P \cap R, \tag{2.1}$$

$$\forall\, g \in G \backslash PR \quad \exists\, p \in P \quad \exists\, r \in R, \tag{2.2}$$

such that $pg = gr$ and $\pi(p) \neq \rho(r)$, and put

$$e = \sum_{p \in P;\, r \in R} \pi(p)\rho(r)pr. \tag{2.3}$$

Then $(KG)e$ is a minimal left ideal of the group algebra KG; even more $(KG)e$ is an absolutely irreducible KG-module.

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[1] The proofs in this section are all imitated from [1, §28].

The proof relies upon the following

LEMMA 2.2. $\{x\in KG;\ pxr=\pi(p^{-1})\rho(r^{-1})x,\ \ \forall p\in P,\ \ \forall r\in R\}=Ke.$

PROOF. If $x=\sum_{g\in G} k(g)g$, $k(g)\in K$, is an element of KG, satisfying $pxr=\pi(p^{-1})\rho(r^{-1})x$, $\forall p\in P$, $\forall r\in R$, then elementary considerations, using (2.1) and (2.2), show that $x=[P\cap R:1]^{-1}k(1)e$. The other inclusion is still more trivial.

PROOF OF THEOREM 2.1. Using the lemma we get $e^2=\gamma e$ for some $\gamma\in K$. In order to show $\gamma\neq 0$, we introduce $T\in \mathrm{Hom}_K(KG, KG)$ by the definition: $Tx=xe$ for all $x\in KG$; considering the matrix of T with respect to the basis $\{g; g\in G\}$, we get $\mathrm{trace}(T)=[G:1][P\cap R:1]$; if we let $\{v_1, v_2, \cdots, v_{[G:1]}\}$ be a basis of KG with $\{v_1, v_2, \cdots, v_f\}$ a basis of the subspace $(KG)e$ we easily get $\mathrm{trace}(T)=f\gamma$; a comparison shows $\gamma=f^{-1}[G:1][P\cap R:1]\neq 0$. We now put $u=\gamma^{-1}e$; then $u^2=u$ is an idempotent element of KG, and in order to show that $(KG)e=(KG)u$ is a minimal left ideal of KG, we just have to prove that $u(KG)u$ is a division ring (see e.g. [1, (25.11)]). However, using Lemma 2.2, it is elementary to check that $u(KG)u=e(KG)e=Ku\cong K$. *The fact that* $(KG)e$ is absolutely irreducible as a KG-module follows from the isomorphism $\mathrm{Hom}_{KG}((KG)u, (KG)u)\cong u(KG)u\cong K$ (see e.g. [1, (26.7) and (29.13)]).

COROLLARY 2.3. $((KG)e:K)=\gamma^{-1}[G:1][P\cap R:1]$.

REMARK. If P and R are fixed, then a pair (π, ρ) of linear characters satisfying (2.1) and (2.2) will be called an *admissible pair*. The element e, defined in (2.3), will be called the *Young element* associated with the pair, and it will be denoted $e_{\pi\rho}$.

3. **Equivalence of Young elements.** We keep P, R, and G fixed, letting (π, ρ) vary over all the admissible pairs. In order to get information on the products $e_{\pi\rho}e_{\pi'\rho'}$ *we assume throughout this section and the next one that* $R\subseteq N(P)$; then $H=PR=RP$ is a group, and a homomorphism $\phi: R\to \mathrm{Aut}(P)$ is given by $(\phi(r))(p)=rpr^{-1}$. If $\pi: P\to K$ is a linear character, then so is $\pi\phi(r)$ for any $r\in R$, and we may introduce an equivalence-relation $\sim$ in the set of all linear characters $P\to K$ by putting $\pi\sim\pi'$ if and only if there is an $r\in R$ with $\pi'=\pi\phi(r)$. We also introduce the subsets $R_{\pi\pi'}$ of R by $R_{\pi\pi'}=\{r\in R;\ \pi'=\pi\phi(r)\}$; $R_{\pi\pi}$ (which shall be denoted just R_π) is then a subgroup of R, and $\pi\sim\pi'$ if and only if $R_{\pi\pi'}\neq\varnothing$. These facts are all immediate consequences of the definitions; so is also

LEMMA 3.1. *Let* $\pi\sim\pi'$ $(\pi'=\pi\phi(r)$, *say*); *then if* (π, ρ) *is admissible so is also* (π', ρ), *and*

$$R_\pi = rR_{\pi'}r^{-1}, \qquad R_{\pi\pi'} = R_\pi r = rR_{\pi'}.$$

We now have (using first shift of variables, then the definition of (,),[2] and, finally, the fact that $(\pi, \pi'\phi(r))=1$ if $\pi=\pi'\phi(r)$, 0 if $\pi\neq\pi'\phi(r)$)

$$\begin{aligned} e_{\pi\rho}e_{\pi'\rho'} &= \sum_{p\in P;r\in R} \pi(p)\rho(r)\left[\sum_{p'\in P;r'\in R} \pi'(p')\rho'(r')pr p'r'\right] \\ &= \sum_{p\in P;r\in R} \pi(p)\rho(r)[\pi'\phi(r^{-1})](p^{-1})\rho'(r^{-1}) \\ &\quad \cdot\left[\sum_{p''\in P;r''\in R} [\pi'\phi(r^{-1})]('p')\rho'(r'')p''r''\right] \\ &= [P:1]\sum_{r\in R} (\pi, \pi'\phi(r^{-1}))\rho(r)\rho'(r^{-1})e_{\pi'\phi(r^{-1}),\rho'} \\ &= [P:1]\sum_{r\in R_{\pi\pi'}} \rho(r)\rho'(r^{-1})e_{\pi\rho'}. \end{aligned}$$

Hence, if $R_{\pi\pi'}=\varnothing$ (i.e. if $\pi\not\sim\pi'$) then $e_{\pi\rho}e_{\pi'\rho'}=0$, whereas, if $\pi\sim\pi'$ ($\pi'=\pi\phi(r_0)$, say) then

$$\begin{aligned} e_{\pi\rho}e_{\pi'\rho'} &= [P:1]\sum_{r\in r_0R_{\pi'}} \rho(r)\rho'(r^{-1})e_{\pi\rho'} \\ &= [P:1]\sum_{r\in R_{\pi'}} \rho(r_0)\rho(r)\rho'(r_0^{-1})\rho'(r^{-1})e_{\pi\rho'} \\ &= [P:1][R_{\pi'}:1](\rho|R_{\pi'}, \rho'|R_{\pi'})\rho(r_0)\rho'(r_0^{-1})e_{\pi\rho'}. \end{aligned}$$

Altogether we have proved

LEMMA 3.2.

$$\begin{aligned} e_{\pi\rho}e_{\pi'\rho'} &= 0, \quad \textit{if } \pi\not\sim\pi' \quad (\textit{i.e. if } R_{\pi\pi'} = \varnothing) \\ &= [P:1][R_{\pi'}:1](\rho|R_{\pi'}, \rho'|R_{\pi'})\rho(r_0)\rho'(r_0^{-1})e_{\pi\rho'}, \\ &\qquad\qquad \textit{if } R_{\pi\pi'} = r_0R_{\pi'} \quad (\neq\varnothing) \end{aligned}$$

from which we get (using also Corollary 2.3)

COROLLARY 3.3. $e_{\pi\rho}e_{\pi'\rho'}\neq 0$ *if and only if* $\pi\sim\pi'$ *and* $\rho|R_{\pi'}=\rho'|R_{\pi'}$.

COROLLARY 3.4. $\gamma_{\pi\rho}=[P:1][R_\pi:1]$, *where* $e_{\pi\rho}^2=\gamma_{\pi\rho}e_{\pi\rho}$.

COROLLARY 3.5. $((KG)e_{\pi\rho}:K)=[G:1][P\cap R:1][P:1]^{-1}[R_\pi:1]^{-1}$.

Now, it is well known that the KH-modules $(KH)e_{\pi\rho}$ and $(KH)e_{\pi'\rho'}$

[2] Here the symbol (,) denotes "inner product" of characters; elsewhere we use it to denote "greatest common divisor."

are isomorphic if and only if $e_{\pi\rho}(KH)e_{\pi'\rho'}\neq 0$ (see e.g. [**1**, (25.13)]). Since every element of H is of the form rp, $r\in R$, $p\in P$, we easily see that $e_{\pi\rho}e_{\pi'\rho'}=0$ implies $e_{\pi\rho}(KH)e_{\pi'\rho'}=0$; the opposite implication is trivial, so we have

LEMMA 3.6. *$e_{\pi\rho}$ and $e_{\pi'\rho'}$ are equivalent within KH (i.e. $(KH)e_{\pi\rho}$ and $(KH)e_{\pi'\rho'}$ are isomorphic as KH-modules) if and only if $e_{\pi\rho}e_{\pi'\rho'}\neq 0$.*

This together with Lemma 3.2 constitutes a proof of

THEOREM 3.7. *$e_{\pi\rho}$ and $e_{\pi'\rho'}$ are equivalent within KH if and only if $\pi\sim\pi'$ and $\rho|R_{\pi'}=\rho'|R_{\pi'}$.*

REMARK. Of course the condition $\rho|R_{\pi'}=\rho'|R_{\pi'}$ may be replaced by $\rho|R_{\pi}=\rho'|R_{\pi}$ (if $\pi\sim\pi'$ then R_π and $R_{\pi'}$ are conjugate subgroups of R).

4. Representations afforded by Young elements. We keep the notation from §3, and start by giving a basis of the vector space $(KH)e_{\pi\rho}$.

THEOREM 4.1. *The set $B_{\pi\rho}=\{e_{\pi'\rho};\ \pi'\sim\pi\}$ is a basis of $(KH)e_{\pi\rho}$.*

PROOF. Using the formulae

$$se_{\pi\rho} = \rho(s^{-1})e_{\pi\phi(s^{-1}),\rho}, \qquad s\in R, \tag{4.1}$$

$$qe_{\pi\rho} = \pi(q^{-1})e_{\pi\rho}, \qquad q\in P, \tag{4.2}$$

it is easy to see that $B_{\pi\rho}$ spans $(KH)e_{\pi\rho}$ as a vector space. By Corollary 3.5 we have $((KH)e_{\pi\rho}:K)=[H:1][P\cap R:1][P:1]^{-1}[R_\pi:1]^{-1}=[R:R_\pi]$, which is easily seen to be $\geqq$ the number of elements of $B_{\pi\rho}$; hence the result follows.

COROLLARY 4.2. *If $\pi\sim\pi'$ then $e_{\pi\rho}=e_{\pi'\rho}$ if and only if $\pi=\pi'$.*

PROOF. If the corollary did not hold, we should get less than $[R:R_\pi]$ elements in $B_{\pi\rho}$.

THEOREM 4.3. *Let $H'=\{h\in H;\ he_{\pi\rho}\in Ke_{\pi\rho}\}$; then $H'=PR_\pi$, and $(KH)e_{\pi\rho}\cong KH\otimes_{KH'}(KH')e_{\pi\rho}$ as KH-modules.*

PROOF. $H'=PR_\pi$ is a trivial consequence of (4.1)-(4.2) and Theorem 4.1. The desired isomorphism is given by the map

$$\Big(\sum_{h\in H}\alpha(h)h\otimes\sum_{h'\in H'}\beta(h')h'e_{\pi\rho}\Big)\to\Big(\sum_{h\in H;\,h'\in H'}\alpha(h)\beta(h')hh'e_{\pi\rho}\Big),$$

which is clearly well defined and epic; monicness follows by counting dimensions.

This theorem tells us that representations afforded by Young-elements are induced monomial representations (see [1, p. 314]); therefore, in some sense, Young elements cannot give any new information concerning the representations; however, in some cases they may be of aid, because they tell us to which subgroups, and to which characters of these subgroups we should apply induction in order to get at (some of) the induced representations. Also it is an advantage that the method only yields *irreducible* representations.

5. **Representations of metacyclic groups.** In this section we prove that for a metacyclic group G the Young-elements will yield a full set of nonisomorphic, irreducible left ideals of CG. Let G be given by generators a and b and relations $a^m=1$, $b^s=a^t$, $bab^{-1}=a^r$, where m, s, t, and r are integers, satisfying $(m, r)=1$, $t(r-1)\equiv r^s-1\equiv 0 \pmod{m}$, $m>0$, $r>0$, $s>0$, and $t\geqq 0$.

We put $P=\{a\}$, $R=\{b\}$; then $R\subseteq N(P)$, $H=PR=G$, R is of order $n=ms/(m, t)$, and ϕ is given by $(\phi(b^\nu))(a^\mu)=a^{\mu r^\nu}$. Letting ζ denote a primitive mth root of unity, η a primitive nth root of unity, we have m linear representations π_i, $i=0, 1, \cdots, m-1$, of P and n linear representations ρ_j, $j=0, 1, \cdots, n-1$, of R, given by $\pi_i(a)=\zeta^i$, $\rho_j(b)=\eta^j$. Since $P\cap R=\{a^t\}=\{b^s\}$ the pair (π_i, ρ_j) is admissible if and only if $\zeta^{it}=\eta^{js}$; the solution of this equation depends on the choice of ζ and η; if we choose ζ and η such that $\zeta^{(m,t)}=\eta^s$ (which can certainly be done) we get: (π_i, ρ_j) is admissible if and only if j is of the form $j=it/(m, t)+qm/(m, t)$ with $-it/m\leqq q<s-it/m$ (the last condition just ensures that j is in the interval from 0 to $n-1$).

We shorten the notation by putting $R_{\pi_i}=R_i$ (and $e_{\pi_i\rho_j}=e_{ij}$); then R_i is of the form $\{b^{s_i}\}$, where s_i is the least positive integer with $\pi_i=\pi_i\phi(b^{s_i})$, that is: s_i is the minimal positive, integral solution of the congruence $i\equiv ir^x \pmod{m}$; therefore s_i is the least natural number with $r^x\equiv 1 \pmod{m/(m, i)}$; since $r^s\equiv 1 \pmod{m/(m, i)}$, we must have $s_i | s$.

Let now i be fixed $(0\leqq i\leqq m-1)$, and let q, q' be integers satisfying $-it/m\leqq q$, $q'<s-it/m$, such that (putting $j=it/(m, t)+qm/(m, t)$ and $j'=it/(m, t)+q'm/(m, t)$) e_{ij} and $e_{ij'}$ are Young-elements. Then e_{ij} and $e_{ij'}$ are equivalent if and only if $\rho_j | R_i=\rho_{j'} | R_i$, i.e. if and only if $js_i\equiv j's_i \pmod{n}$; this condition is easily seen to be equivalent to the condition $q\equiv q' \pmod{s/s_i}$. Hence among the elements e_{ij}, $j=it/(m, t)+qm/(m, t)$, $-it/m\leqq q<s-it/m$, there are exactly s/s_i pairwise nonequivalent Young-elements; each of these is of degree $[R: 1][R_i: 1]^{-1}=n(n/(n, s_i))^{-1}=(n, s_i)=(ms/(m, t), s_i)=s_i$; their total contribution to "the sum of the squares of the degrees of the

pairwise nonequivalent, irreducible representations of G'' is, therefore, equal to $(s/s_i)\cdot s_i^2 = ss_i$.

Next let i, i', and j be fixed; then e_{ij} is equivalent to $e_{i'j'}$ for some j' if and only if $\pi_i \sim \pi_{i'}$. Therefore, in order to get the "sum of the squares of the degrees of the pairwise nonequivalent, irreducible representations afforded by Young elements," we have to extend the sum $\sum ss_i$ over one representative of each equivalence class $\tilde{\pi}_i$ under $\sim$; however, it is easily seen that the class $\tilde{\pi}_i$ consists of exactly s_i elements, and hence the abovementioned sum equals $\sum_{i=0}^{m-1} s_i^{-1}(ss_i) = ms = [G:1]$. As it is well known this ensures that we have a full set of pairwise nonequivalent, irreducible representations. We collect the results of this section in

THEOREM 5.1. *Let G be a metacyclic group with generators a and b, and relations $a^m = 1$, $b^s = a^t$, $bab^{-1} = a^r$, where m, r, s, and t are integers, satisfying $(m, r) = 1$, $t(r-1) \equiv r^s - 1 \equiv 0 \pmod{m}$, $m > 0$, $r > 0$, $s > 0$, $t \geqq 0$. Let s_i be the order of r as an element of the group of units in $Z_{m/(m,i)}$. In $\{0, 1, \cdots, m-1\}$ we define an equivalence relation $\sim$ by: $i \sim i'$ if and only if there is a ν with $i \equiv i'r^\nu \pmod{m}$, and we let I be a set of representatives of the classes modulo $\sim$. For each $i \in I$ we let $\{q_{1,i}, q_{2,i}, \cdots, q_{s/s_i,i}\}$ be a full set of pairwise incongruent integers modulo s/s_i with $-it/m \leqq q_{k,i} < s - it/m$, and we put $j_{k,i} = it/(m, t) + q_{k,i}m/(m, t)$. Finally, we choose a primitive mth root of unity ζ and a primitive nth root of unity η $(n = ms/(m, t))$ such that $\zeta^{(m,t)} = \eta^s$. Then some of the Young-elements are:*

$$c_{i,j_{k,i}} = \sum_{\alpha=0}^{m-1}\sum_{\beta=0}^{n-1} \zeta^{\alpha i}\eta^{\beta j_{k,i}} a^\alpha b^\beta, \qquad k = 1, 2, \cdots, s/s_i, \qquad i \in I,$$

and

$$\{(CG)e_{i,j_{k,i}};\quad k = 1, 2, \cdots, s/s_i,\quad i \in I\}$$

is a full set of pairwise nonequivalent, irreducible left CG-modules. The representation T_{ik} afforded by $(CG)e_{i,j_{k,i}}$ is of degree s_i and it is induced from the linear representation $\overline{T}_{ik}: G_i \to C$ where G_i is the subgroup of G generated by the elements a and b^{s_i}, and where the action of $\overline{T}_{ik}$ is given by the formulae: $\overline{T}_{ik}(a) = \zeta^{-i}$, $\overline{T}_{ik}(b^{s_i}) = \eta^{-s_i j_{k,i}}$.

The last part of the theorem follows immediately from Theorem 4.1 and Formulae (4.1)–(4.2).

REFERENCE

1. C. W. Curtis and Irving Reiner, *Representation theory of finite groups and associative algebras*, Pure and Appl. Math., vol 11, Interscience, New York, 1962.

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