ON STARLIKE FUNCTIONS
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I. Let $S_k$ be the class of functions

$$f(z) = z + \sum_{n=k+1}^{\infty} a_n z^n,$$

regular and univalent in $|z| < 1$; let $S_k$ be the class of functions

$$f(z) = z + \sum_{n=k+1}^{\infty} a_n z^{-n}, \quad f(z) \neq 0,$$

regular and univalent in $|z| > 1$, except for the simple pole at infinity. Finally, let $S_k^*$ and $S_k^*$ be the corresponding subclasses of starlike functions.

In a recent paper [6], T. H. MacGregor proved that if $f \in S_k^*$ then

$$|a_n| \leq \frac{mk}{n-1} \left| \binom{-2/k}{m} \right|$$

for $mk+1 \leq n \leq (m+1)k$, $m = 1, 2, \ldots$, thus generalizing Golusin's result [2] that if $f(z) = z + \sum_{m=1}^{\infty} a_{mk+1} z^{mk+1}$ belongs to $S_k^*$ then

$$|a_{mk+1}| \leq \left| \binom{-2/k}{m} \right| .$$

It is the purpose of this paper to extend, in Theorem 1, MacGregor's result and then, in §II, to obtain some interesting corollaries to Theorem 1.

The proof of Theorem 1 follows MacGregor's method of proof [6].

**Theorem 1.** Suppose $f \in S_k^*$ and for integral $t \geq 1$ let

$$f(z)^{-t} = z^{-t} + \sum_{r=t+k}^{\infty} a_r (-t)^r z^r .$$

Then

$$|a_r (-t)| \leq \frac{mk}{v+t} \left( \frac{2t/k}{m} \right).$$

Received by the editors January 31, 1967.

1 This research was supported in part by the National Science Foundation under grant NSF-GP-817.
for $-t+mk \leq \nu \leq -t+(m+1)k-1$, $m$ a positive integer, such that $t-(m-1)k \geq 0$.

**Proof.** Let

(1) \[ g(z) = \frac{1}{-t} \frac{zf'(z)}{f(z)} = \frac{zf'(z)}{f(z)} \]

and define

(2) \[ w(z) = \frac{g(z) - 1}{g(z) + 1} = \sum_{\nu=k}^{\infty} \omega_{\nu} z^{\nu}. \]

From (1) and (2) we have

(3) \[ \{zf'(z)^{-t} - t f(z)^{-t}\} w(z) = z[f(z)^{-t}]' + tf(z)^{-t}. \]

Comparing coefficients in (3), we have

(4) \[ -2t \omega_{\nu} = \nu a_{(-t)}^{(-t)}, \quad \nu = k, \ldots, 2k-1. \]

Since $\text{Re}(g(z)) > 0$ in $|z| < 1$, $|w(z)| < 1$ in $|z| < 1$ and thus $\sum_{\nu=k}^{\infty} |\omega_{\nu}|^2 \leq 1$. Therefore, using (4), we have

(5) \[ \sum_{\nu=k}^{2k-1} \nu^2 |a_{\nu+\nu}^{(-t)}|^2 = \sum_{\nu=k}^{2k-1} 4t^2 |\omega_{\nu}|^2 \leq 4t^2 \sum_{\nu=k}^{\infty} |\omega_{\nu}|^2 \leq 4t^2. \]

Writing (3) in the form

\[ \sum_{\nu=-t+k}^{\nu} (\nu + t) a_{\nu}^{(-t)} z^\nu + \sum_{\nu=-t+k}^{\infty} c_{\nu} z^\nu = w(z) \left[ -2t z^{-t} + \sum_{\nu=-t+k}^{\nu} (\nu - t) a_{\nu}^{(-t)} z^\nu \right], \]

multiplying each side of this by its conjugate, integrating around $|z| = r < 1$ and letting $r \to 1$ yields the inequality

\[ \sum_{\nu=-t+k}^{\nu} (\nu + t)^2 |a_{\nu}^{(-t)}|^2 \leq 4t^2 + \sum_{\nu=-t+k}^{\nu} (\nu - t)^2 |a_{\nu}^{(-t)}|^2, \]

from which we obtain

(6) \[ \sum_{\nu=-t+k+1}^{\nu} (\nu + t)^2 |a_{\nu}^{(-t)}|^2 \leq 4t^2 + \sum_{\nu=-t+k}^{\nu} -4vt |a_{\nu}^{(-t)}|^2. \]

In (6) let $p = -t-1+(q+1)k$ and it then becomes
\[
\sum_{r=-t+qk}^{-t+(p+1)k-1} (\nu + t) \left| a_r^{(-t)} \right|^2 \leq 4t^2 + \sum_{r=-t+k}^{-t+qk-1} -4\nu t \left| a_r^{(-t)} \right|^2
\]

\[
= 4t \left[ t + \sum_{r=-t+k}^{-t+qk-1} -\nu \left| a_r^{(-t)} \right|^2 \right]
\]

\[
= 4t \left[ t + \sum_{m=1}^{q-1} \sum_{r=-t+mk}^{-t+(m+1)k-1} -\nu \left| a_r^{(-t)} \right|^2 \right]
\]

Now

\[
\sum_{r=-t+k}^{-t+2k-1} -\nu \left| a_r^{(-t)} \right|^2 = \frac{t - k}{k^2} \sum_{r=-t+k}^{-t+2k-1} \frac{k^2(-\nu)}{t - k} \left| a_r^{(-t)} \right|^2
\]

\[
\leq \frac{t - k}{k^2} \sum_{r=-t+k}^{-t+2k-1} (\nu + t)^2 \left| a_r^{(-t)} \right|^2
\]

\[
\leq (t - k) \frac{4t^2}{k^2} = (t - k) \left( \frac{2t/k}{1} \right)^2,
\]

where the last inequality follows from (5). And we assume

\[
\sum_{r=-t+mk}^{-t+(m+1)k-1} -\nu \left| a_r^{(-t)} \right|^2 \leq (t - mk) \left( \frac{2t/k}{m} \right)^2, \quad m = 2, \ldots, q - 1.
\]

Then from (7)

\[
\sum_{r=-t+qk}^{-t+(q+1)k-1} (\nu + t)^2 \left| a_r^{(-t)} \right|^2 \leq 4t \left[ t + \sum_{m=1}^{q-1} \sum_{r=-t+mk}^{-t+(m+1)k-1} -\nu \left| a_r^{(-t)} \right|^2 \right]
\]

\[
\leq 4t \left[ t + \sum_{m=1}^{q-1} (t - mk) \left( \frac{2t/k}{m} \right)^2 \right]
\]

\[
= (qk)^2 \left( \frac{2t/k}{m} \right)^2, \quad q \geq 2,
\]

so long as \( t - (q - 1)k \geq 0 \). From this it follows that

\[
\sum_{r=-t+qk}^{-t+(q+1)k-1} -\nu \left| a_r^{(-t)} \right|^2 = \frac{t - qk}{(qk)^2} \sum_{r=-t+qk}^{-t+(q+1)k-1} \frac{(qk)^2(-\nu)}{t - qk} \left| a_r^{(-t)} \right|^2
\]

\[
\leq \frac{t - qk}{(qk)^2} \sum_{r=-t+qk}^{-t+(q+1)k-1} (\nu + t)^2 \left| a_r^{(-t)} \right|^2
\]

\[
\leq (t - qk) \left( \frac{2t/k}{q} \right)^2.
\]
Thus, by induction, we prove that

\begin{equation}
\sum_{\nu=-t+mk}^{-t+(m+1)k-1} (\nu + t)^2 |a_r(\nu)|^2 \leq (mk)^2 \left(\frac{2t/k}{m}\right)^2
\end{equation}

and

\begin{equation}
\sum_{\nu=-t+mk}^{-t+(m+1)k-1} -\nu |a_r(\nu)|^2 \leq (t - mk) \left(\frac{2t/k}{m}\right)^2
\end{equation}

for \( m \) such that \( t - (m-1)k \geq 0 \).

The theorem follows from (8).

If \( \nu = -t+mk \) the inequality of Theorem 1 is sharp: consider \( f(z) = z/(1+z^k)^{2/k} \). Also if \( -t+k < \nu \leq -t+2k-1 \) the inequality is sharp: consider \( f(z) = z/(1+z^{k+1})^{2/(k+1)} \).

II. For \( f \in S_k \) let \( G = f(|z| < 1) \). Suppose \( D \) is the image of \( G \) under the transformation \( w = 1/\xi \) and let \( E \) be the complement of \( D \) in the \( w \)-plane. \( E \) is a bounded continuum of capacity one with the coordinate system origin \( 0 \in E \). We define the moments \( s_n = \int w^n d\mu, \ n = 1, 2, \cdots \), where \( \mu \) is the natural mass distribution on \( E \). The coefficients of \( f \) can be expressed as polynomials in \( s_n \) [5]. Let \( \tilde{f}(z) = 1/f(1/z) \) (\( \tilde{f} \) maps \( |z| > 1 \) conformally onto \( D \)) and let \( \phi(w) = w + \sum_{n=k+1}^\infty b_n w^n \) and \( \phi(w) = w + \sum_{n=k-1}^\infty \tilde{b}_n w^n \) be the inverse functions to \( f \) and \( \tilde{f} \) respectively.

It has been proved ([3], [4], [7]) that

1. \( b_n = (1/n) a_{-n-1}^{(-n)} \), \( n = 1, 2, \cdots \).
2. \( \tilde{b}_n = -(1/n) a_1^{(-n)} \), \( n = 1, 2, \cdots \).
3. \( s_n = a_0^{(-n)} \), \( n = 1, 2, \cdots \).
4. If \( s_n(\bar{0}) \) are the moments obtained when the origin is situated at the center of gravity \( \bar{0} \) (= \( s_1 \) in terms of \( 0 \)) of \( E \), i.e., the moments with respect to the center of gravity \( \bar{0} \) of \( E \), then

\[ s_n = s_n(\bar{0}) + \binom{n}{1} s_{n-1}(\bar{0}) s_1 + \binom{n}{2} s_{n-2}(0) s_1^2 + \cdots + s_1. \]

In particular, if \( s_1 = 0 \) then \( s_n(\bar{0}) = s_n \).

**Theorem 2.** If \( \phi(w) = w + \sum_{n=k+1}^\infty b_n w^n \in S_k^{*1} \), then

\[ |b_n| \leq \frac{mk}{n(n-1)} \left(\frac{2n/k}{m}\right)^2 \]

for \( mk + 1 \leq n \leq (m+1)k, \ m = 1, 2, \cdots \).

**Theorem 3.** If \( \tilde{\phi}(w) = w + \sum_{n=k-1}^\infty \tilde{b}_n w^{-n} \in \sum_k^{*1} \) then
\[ |b_n| \leq \frac{m k}{n(n+1)} \binom{2n/k}{m} \]

for \( mk - 1 \leq n \leq (m+1)k - 2, m = 1, 2, \ldots \).

**Theorem 4.** If \( f \in S_k^* \) then

\[ |s_n| \leq \frac{m k}{n} \binom{2n/k}{m} \]

for \( mk \leq n \leq (m+1)k - 1, m = 1, 2, \ldots \).

These theorems follow as corollaries to Theorem 1 and 1, 2 and 3 above, respectively.

If \( f \in S_k^* \), i.e., \( f \) is starlike and \( a_2 = 0 \), then \( s_1 = 0 \). In this case \( s_n(0) = s_n \) and we have the following estimates for \( s_n(0) \) as a corollary to Theorem 4.

**Corollary.** If \( f \in S_k^* \) then

\[ |s_{2m}(0)| \leq \binom{2m}{m}, \quad |s_{2m+1}(0)| \leq \frac{2m}{2m+1} \binom{2m+1}{m} \]

for \( m = 1, 2, \ldots \).

The estimates (10) hold for a larger class of functions.

**Definition.** Let \( \overline{S}^* \) denote the class of univalent functions (functions in \( S_1 \)) such that \( f \in \overline{S}^* \) if and only if the corresponding continuum \( E \) contains its center of gravity \( \overline{0} \) and is star shaped with respect to \( \overline{0} \).

**Theorem 5.** If \( f \in \overline{S}^* \) then the estimates (10) hold.

**Proof.** Suppose \( f \in \overline{S}^* \) and has corresponding continuum \( E \). Let \( E_0 = \{ w' | w' = w - \overline{0}, w \in E \} \) and note that \( E_0 \) has its center of gravity at the coordinate system origin and is starshaped with respect to the origin. There exists a function \( f_0 \in S_1 \) such that its corresponding continuum is \( E_0 \). \( f_0 \in S_2 \) since the center of gravity of \( E_0 \) is \( w = 0 \) which implies \( a_2 = 0 \). Since \( f_0(s^{-1})^{-1} \) maps \( |z| > 1 \) onto the complement of \( E_0 \) and \( E_0 \) is starshaped with respect to \( w = 0, f_0(s^{-1})^{-1} \in \Sigma_2^* \). \( f_0(s^{-1})^{-1} \in \Sigma_2^* \Rightarrow f_0 \in S_2^* \). Thus (10) holds for \( f_0 \). But \( s_n(0) \) for \( E \) is identical with \( s_n(0) = s_n \) for \( E_0 \), since \( E_0 \) is a translation of \( E \). This completes the proof.

The estimates (10) are sharp for \( s_{2m}(0), m = 1, 2, \ldots \), and \( s_3(0) \) (see the comments on the sharpness of Theorem 1).

Since it is known [3] that \( |s_2(0)| \leq 2 \) for all \( f \in S_1 \), it seems reasonable to conjecture that
\[ |s_{2m}(0)| \leq \binom{2m}{m}, \quad m = 1, 2, \ldots \]

for all \( f \in S_1 \). It is also known [1] that \( |s_3(0)| = | -3a_4 + 6a_2a_3 - 3a_2^2 | \leq 2 \) for all \( f \in S_1 \).

References


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