ON STARLIKE FUNCTIONS[1]

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I. Let S_k be the class of functions

$$f(z) = z + \sum_{n=k+1}^{\infty} a_n z^n,$$

regular and univalent in $|z| < 1$; let Σ_k be the class of functions

$$\tilde{f}(z) = z + \sum_{n=k-1}^{\infty} a_n z^{-n}, \quad f(z) \neq 0,$$

regular and univalent in $|z| > 1$, except for the simple pole at infinity. Finally, let S_k^* and Σ_k^* be the corresponding subclasses of starlike functions.

In a recent paper [6], T. H. MacGregor proved that *if* $f \in S_k^*$ *then*

$$|a_n| \leqq \frac{mk}{n-1} \left| \binom{-2/k}{m} \right|$$

for $mk+1 \leqq n \leqq (m+1)k$, $m = 1, 2, \cdots$, thus generalizing Golusin's result [2] that *if* $f(z) = z + \sum_{m=1}^{\infty} a_{mk+1} z^{mk+1}$ *belongs to* S_k^* *then*

$$|a_{mk+1}| \leqq \left| \binom{-2/k}{m} \right|.$$

It is the purpose of this paper to extend, in Theorem 1, MacGregor's result and then, in §II, to obtain some interesting corollaries to Theorem 1.

The proof of Theorem 1 follows MacGregor's method of proof [6].

THEOREM 1. *Suppose* $f \in S_k^*$ *and for integral* $t \geqq 1$ *let*

$$f(z)^{-t} = z^{-t} + \sum_{\nu=-t+k}^{\infty} a_\nu^{(-t)} z^\nu.$$

Then

$$|a_\nu^{(-t)}| \leqq \frac{mk}{\nu + t} \binom{2t/k}{m}$$

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for $-t+mk \leqq \nu \leqq -t+(m+1)k-1$, *m a positive integer, such that* $t-(m-1)k \geqq 0$.

PROOF. Let

$$g(z) = \frac{1}{-t}\,\frac{z[f(z)^{-t}]'}{f(z)^{-t}} = \frac{zf'(z)}{f(z)} \tag{1}$$

and define

$$w(z) = \frac{g(z)-1}{g(z)+1} = \sum_{\nu=k}^{\infty} \omega_\nu z^\nu. \tag{2}$$

From (1) and (2) we have

$$\{z[f(z)^{-t}]' - tf(z)^{-t}\}w(z) = z[f(z)^{-t}]' + tf(z)^{-t}. \tag{3}$$

Comparing coefficients in (3), we have

$$-2t\omega_\nu = \nu a_{-t+\nu}^{(-t)}, \qquad \nu = k, \cdots, 2k-1. \tag{4}$$

Since Re $(g(z))>0$ in $|z|<1$, $|w(z)|<1$ in $|z|<1$ and thus $\sum_{\nu=k}^{\infty}|\omega_\nu|^2 \leqq 1$. Therefore, using (4), we have

$$\sum_{\nu=k}^{2k-1} \nu^2 |a_{-t+\nu}^{(-t)}|^2 = \sum_{\nu=k}^{2k-1} 4t^2 |\omega_\nu|^2 \leqq 4t^2 \sum_{\nu=k}^{\infty} |\omega_\nu|^2 \leqq 4t^2. \tag{5}$$

Writing (3) in the form

$$\sum_{\nu=-t+k}^{p} (\nu+t)a_\nu^{(-t)} z^\nu + \sum_{\nu=p+1}^{\infty} c_\nu z^\nu$$
$$= w(z)\left[-2tz^{-t} + \sum_{\nu=-t+k}^{p-k} (\nu-t)a_\nu^{(-t)} z^\nu\right],$$

multiplying each side of this by its conjugate, integrating around $|z|=r<1$ and letting $r\to 1$ yields the inequality

$$\sum_{\nu=-t+k}^{p} (\nu+t)^2 |a_\nu^{(-t)}|^2 \leqq 4t^2 + \sum_{\nu=-t+k}^{p-k} (\nu-t)^2 |a_\nu^{(-t)}|^2,$$

from which we obtain

$$\sum_{\nu=p-k+1}^{p} (\nu+t)^2 |a_\nu^{(-t)}|^2 \leqq 4t^2 + \sum_{\nu=-t+k}^{p-k} -4\nu t |a_\nu^{(-t)}|^2. \tag{6}$$

In (6) let $p=-t-1+(q+1)k$ and it then becomes

$$\begin{aligned}\sum_{\nu=-t+qk}^{-t+(q+1)k-1}(\nu+t)^2\left|a_\nu^{(-t)}\right|^2 &\leqq 4t^2+\sum_{\nu=-t+k}^{-t+qk-1}-4\nu t\left|a_\nu^{(-t)}\right|^2\\ &= 4t\left[t+\sum_{\nu=-t+k}^{-t+qk-1}-\nu\left|a_\nu^{(-t)}\right|^2\right] \qquad (7)\\ &= 4t\left[t+\sum_{m=1}^{q-1}\sum_{\nu=-t+mk}^{-t+(m+1)k-1}-\nu\left|a_\nu^{(-t)}\right|^2\right].\end{aligned}$$

Now

$$\begin{aligned}\sum_{\nu=-t+k}^{-t+2k-1}-\nu\left|a_\nu^{(-t)}\right|^2 &= \frac{t-k}{k^2}\sum_{-t+k}^{-t+2k-1}\frac{k^2(-\nu)}{t-k}\left|a_\nu^{(-t)}\right|^2\\ &\leqq \frac{t-k}{k^2}\sum_{-t+k}^{-t+2k-1}(\nu+t)^2\left|a_\nu^{(-t)}\right|^2\\ &\leqq (t-k)\frac{4t^2}{k^2}=(t-k)\binom{2t/k}{1}^2,\end{aligned}$$

where the last inequality follows from (5). And we assume

$$\sum_{\nu=-t+mk}^{-t+(m+1)k-1}-\nu\left|a_\nu^{(-t)}\right|^2 \leqq (t-mk)\binom{2t/k}{m}^2, \quad m=2,\cdots,q-1.$$

Then from (7)

$$\begin{aligned}\sum_{\nu=-t+qk}^{-t+(q+1)k-1}(\nu+t)^2\left|a_\nu^{(-t)}\right|^2 &\leqq 4t\left[t+\sum_{m=1}^{q-1}\sum_{\nu=-t+mk}^{-t+(m+1)k-1}-\nu\left|a_\nu^{(-t)}\right|^2\right]\\ &\leqq 4t\left[t+\sum_{m=1}^{q-1}(t-mk)\binom{2t/k}{m}^2\right]\\ &= (qk)^2\binom{2t/k}{m}^2, \quad q\geqq 2,\end{aligned}$$

so long as $t-(q-1)k\geqq 0$. From this it follows that

$$\begin{aligned}\sum_{\nu=-t+qk}^{-t+(q+1)k-1}-\nu\left|a_\nu^{(-t)}\right|^2 &= \frac{t-qk}{(qk)^2}\sum_{-t+qk}^{-t+(q+1)k-1}\frac{(qk)^2(-\nu)}{t-qk}\left|a_\nu^{(-t)}\right|^2\\ &\leqq \frac{t-qk}{(qk)^2}\sum_{-t+qk}^{-t+(q+1)k-1}(\nu+t)^2\left|a_\nu^{(-t)}\right|^2\\ &\leqq (t-qk)\binom{2t/k}{q}^2.\end{aligned}$$

Thus, by induction, we prove that

$$\sum_{\nu=-t+mk}^{-t+(m+1)k-1} (\nu+t)^2 \left| a_\nu^{(-t)} \right|^2 \leqq (mk)^2 \binom{2t/k}{m}^2 \tag{8}$$

and

$$\sum_{\nu=-t+mk}^{-t+(m+1)k-1} -\nu \left| a_\nu^{(-t)} \right|^2 \leqq (t-mk) \binom{2t/k}{m}^2 \tag{9}$$

for m such that $t-(m-1)k \geqq 0$.

The theorem follows from (8).

If $\nu=-t+mk$ the inequality of Theorem 1 is sharp: consider $f(z)=z/(1+z^k)^{2/k}$. Also if $-t+k<\nu\leqq -t+2k-1$ the inequality is sharp: consider $f(z)=z/(1+z^{\nu+t})^{2/(\nu+t)}$.

II. For $f\in S_k$ let $G=f(|z|<1)$. Suppose D is the image of G under the transformation $w=1/\zeta$ and let E be the complement of D in the w-plane. E is a bounded continuum of capacity one with the coordinate system origin $0\in E$. We define the moments $s_n=\int w^n d\mu$, $n=1, 2, \cdots$, where μ is the natural mass distribution on E. The coefficients of f can be expressed as polynomials in s_n [5]. Let $\tilde{f}(z)=1/f(1/z)$ ($\tilde{f}$ maps $|z|>1$ conformally onto D) and let $\phi(w)=w+\sum_{n=k+1}^{\infty} b_n w^n$ and $\tilde{\phi}(w)=w+\sum_{n=k-1}^{\infty} \tilde{b}_n w^n$ be the inverse functions to f and $\tilde{f}$ respectively.

It has been proved ([3], [4], [7]) that

1. $b_n=(1/n)a_{-1}^{(-n)}$, $n=1, 2, \cdots$.
2. $\tilde{b}_n=-(1/n)a_1^{(-n)}$, $n=1, 2, \cdots$.
3. $s_n=a_0^{(-n)}$, $n=1, 2, \cdots$.
4. If $s_n(\bar{0})$ are the moments obtained when the origin is situated at the center of gravity $\bar{0}$ ($=s_1$ in terms of 0) of E, i.e., the moments with respect to the center of gravity $\bar{0}$ of E, then

$$s_n = s_n(\bar{0}) + \binom{n}{1} s_{n-1}(\bar{0}) s_1 + \binom{n}{2} s_{n-2}(0) s_1^2 + \cdots + s_1^n.$$

In particular, if $s_1=0$ then $s_n(\bar{0})=s_n$.

THEOREM 2. *If* $\phi(w)=w+\sum_{n=k+1}^{\infty} b_n w^n \in S_k^{*-1}$, *then*

$$|b_n| \leqq \frac{mk}{n(n-1)} \binom{2n/k}{m}$$

for $mk+1\leqq n\leqq (m+1)k$, $m=1, 2, \cdots$.

THEOREM 3. *If* $\tilde{\phi}(w)=w+\sum_{n=k-1}^{\infty} \tilde{b}_n w^{-n} \in \sum_k^{*-1}$ *then*

$$|\bar{b}_n| \leqq \frac{mk}{n(n+1)}\binom{2n/k}{m}$$

for $mk-1 \leqq n \leqq (m+1)k-2$, $m=1, 2, \cdots$.

THEOREM 4. *If* $f \in S_k^*$ *then*

$$|s_n| \leqq \frac{mk}{n}\binom{2n/k}{m}$$

for $mk \leqq n \leqq (m+1)k-1$, $m=1, 2, \cdots$.

These theorems follow as corollaries to Theorem 1 and 1, 2 and 3 above, respectively.

If $f \in S_2^*$, i.e., f is starlike and $a_2=0$, then $s_1=0$. In this case $s_n(\bar{0}) = s_n$ and we have the following estimates for $s_n(\bar{0})$ as a corollary to Theorem 4.

COROLLARY. *If* $f \in S_2^*$ *then*

$$(10) \qquad |s_{2m}(\bar{0})| \leqq \binom{2m}{m}, \qquad |s_{2m+1}(\bar{0})| \leqq \frac{2m}{2m+1}\binom{2m+1}{m}$$

for $m=1, 2, \cdots$.

The estimates (10) hold for a larger class of functions.

DEFINITION. *Let* $\bar{S}^*$ *denote the class of univalent functions (functions in* S_1*) such that* $f \in \bar{S}^*$ *if and only if the corresponding continuum* E *contains its center of gravity* $\bar{0}$ *and is starshaped with respect to* $\bar{0}$.

THEOREM 5. *If* $f \in \bar{S}^*$ *then the estimates* (10) *hold.*

PROOF. Suppose $f \in \bar{S}^*$ and has corresponding continuum E. Let $E_0 = \{w' \mid w' = w - \bar{0},\ w \in E\}$ and note that E_0 has its center of gravity at the coordinate system origin and is starshaped with respect to the origin. There exists a function $f_0 \in S_1$ such that its corresponding continuum is E_0. $f_0 \in S_2$ since the center of gravity of E_0 is $w=0$ which implies $a_2=0$. Since $f_0(z^{-1})^{-1}$ maps $|z|>1$ onto the complement of E_0 and E_0 is starshaped with respect to $w=0$, $f_0(z^{-1})^{-1} \in \Sigma_2^*$. $f_0(z^{-1})^{-1} \in \Sigma_2^* \Rightarrow f_0 \in S_2^*$. Thus (10) holds for f_0. But $s_n(\bar{0})$ for E is identical with $s_n(0) = s_n$ for E_0, since E_0 is a translation of E. This completes the proof.

The estimates (10) are sharp for $s_{2m}(\bar{0})$, $m=1, 2, \cdots$, and $s_3(\bar{0})$ (see the comments on the sharpness of Theorem 1).

Since it is known [3] that $|s_2(\bar{0})| \leqq 2$ for all $f \in S_1$, it seems reasonable to conjecture that

$$|s_{2m}(\bar{0})| \leqq \binom{2m}{m}, \qquad m = 1, 2, \cdots$$

for all $f \in S_1$. It is also known [1] that $|s_3(\bar{0})| = |-3a_4 + 6a_2a_3 - 3a_2^3| \leqq 2$ for all $f \in S_1$.

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