LOCALIZATION OF A THEOREM OF GLIMM

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1. **Introduction.** For the terminology and the main results concerning C^*-algebras, we refer to [1] where the reader will find the references. Let us just recall that if H is a Hilbert space, we denote by $\mathcal{LC}(H)$ the set of all compact linear operators in H, and that a C^*-algebra is said to be elementary if it is isomorphic to some algebra $\mathcal{LC}(H)$.

GLIMM'S THEOREM [5, THEOREMS 1 AND 2]. *Let A be a separable C^*-algebra. The following conditions are equivalent*;

(i) *A is postliminar* (=GCR);

(ii) *A is type* I;

(iii) *A has no type* II *representation*;

(iv) *A has no type* III *representation*;

(v) *for every irreducible representation π of A, $\pi(A)$ contains $\mathcal{LC}(H_\pi)$ (H_π, Hilbert space of π)*;

(vi) *if π_1, π_2 are two irreducible representations of A with the same kernel, π_1 and π_2 are (unitarily) equivalent*;

(vii) *the Mackey Borel structure on $\hat{A}$ is standard*;

(viii) *the Mackey Borel structure on $\hat{A}$ is metrically countably separated.*

So, a separable C^*-algebra A is either "very good" or "very bad" according as it is or it is not postliminar. We shall see that the same strong dichotomy occurs if we study the representations of A with kernel a fixed primitive ideal of A. This will sharpen a part of Glimm's theorem.

The proof is a synthesis of known arguments.

2. Let X_0 be the group $Z/2Z$ with two elements 0, 1; we shall denote by X the compact metrizable group $X_0 \times X_0 \times X_0 \times \cdots$, by X' the subgroup of X consisting of those elements with all but a finite number of coordinates equal to 0. Let E be a Borel space. It follows from [5, p. 593, lines 1–11 and p. 596, lines 7–8] that if there exists an injective Borel map from X/X' into E, then E is not metrically countably separated.

3. THEOREM. *Let A be a separable C^*-algebra, J a primitive ideal of A, E the set of elements in $\hat{A}$ with kernel J; we consider on E the Mackey Borel structure.*

(a) *Suppose that A/J has a nonzero elementary closed two-sided ideal.*

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Then E reduces to one point π, and $\pi(A)\supset\mathcal{LC}(H_\pi)$; every factor representation of A with kernel J is type I.

(b) *Suppose that A/J has no nonzero elementary closed two-sided ideal. Then E has the power of the continuum; there exists an injective Borel map from X/X' into E; for every $\pi\in E$ one has $\pi(A)\cap\mathcal{LC}(H_\pi)=0$; A has a type* II *factor representation with kernel J and a type* III *factor representation with kernel J.*

One is reduced at once to the case where $J=0$. So we shall assume that A is primitive.

Suppose that A has a nonzero elementary closed two-sided ideal K. Since A is primitive, there exists a faithful irreducible representation π of A. The restriction of π to K is faithful irreducible [**1**, 2.11.3] whence $\pi(K)=\mathcal{LC}(H_\pi)$ [**1**, 4.1.5] so that $\pi(A)\supset\mathcal{LC}(H_\pi)$. Every faithful irreducible representation of A is equivalent to π [**1**, 4.1.10]. From this and from the reasoning of [**1**, p. 168, (v)$\Rightarrow$(ii)], it results that every faithful factor representation of A is type I. This proves (a).

Suppose that A has no nonzero elementary closed two-sided ideal. Let K be the largest postliminar closed two-sided ideal of A, and let $\pi\in E$. If $K\neq 0$, $\pi|K$ is irreducible [**1**, 2.11.3], whence $\pi(K)\supset\mathcal{LC}(H_\pi)$ [**1**, 4.3.7], and so A contains a nonzero elementary closed two-sided ideal contrary to the hypothesis. So $K=0$, and A is antiliminar (=NGCR). On the other hand, if $\pi(A)\cap\mathcal{LC}(H_\pi)\neq 0$, one has $\pi(A)\supset\mathcal{LC}(H_\pi)$ [**1**, 4.1.10], which is still absurd; this proves that $\pi(A)\cap\mathcal{LC}(H_\pi)=0$.

Since A is separable antiliminar, one can use the proof of [**5**, p. 587–589]. In this proof, Glimm constructs a factor representation ϕ of A which is type II (resp. III) if a certain parameter is equal to $\frac{1}{2}$ (resp. different from $\frac{1}{2}$). By refining this construction a little, we shall show that (under our present hypothesis) ϕ is actually faithful (the following idea was already used in [**3**]). Since A is separable primitive, there exists a decreasing sequence $(J_1, J_2, \cdots)$ of nonzero closed two-sided ideals of A such that every nonzero primitive ideal of A contains one of the J_n [2, Lemma 15]. If A is realized as an irreducible C^*-algebra of operators in a Hilbert space, each J_n operates irreducibly [**1**, 2.11.3]; granted this, we can, in the proof of [**5**, Lemma 4], choose the element C_0 (p. 578) in J_{n+1} whence $V(a_1,\cdots,a_{n+1})\in J_{n+1}$. But, coming back to p. 587 of [**5**], one has

$$\begin{aligned}(\phi(V(a_1,\cdots,a_{n+1})V(a_1,\cdots,a_{n+1})^*)x\mid x)\\ &= h(V(a_1,\cdots,a_{n+1})V(a_1,\cdots,a_{n+1})^*)\\ &= g(V(a_1,\cdots,a_{n+1})V(a_1,\cdots,a_{n+1})^*)\neq 0,\end{aligned}$$

whence $V(a_1, \cdots, a_{n+1}) \notin \text{Ker}\,\phi$, and so $\text{Ker}\,\phi \not\supset J_{n+1}$. This being true for every n, we have $\text{Ker}\,\phi = 0$.

With A antiliminar separable, Glimm constructs in [5, p. 593–596] an injective Borel map of X/X' into $\hat{A}$; with the notations of [5], $f(X/X')$ is equal to K^c (or, more accurately, to the set of classes of elements of K^c). If $\pi \in K^c$, one has $\pi(\sum_{a_1, \cdots, a_n} V(a_1, \cdots, a_n) V(a_1, \cdots, a_n)^*) \neq 0$ [5, p. 593, line 4 from the bottom], and so $\text{Ker}\,\pi \not\supset J_n$. This proves that $\text{Ker}\,\pi = 0$, and so $f(X/X') \subset E$.

From this, we conclude that Card $E \geqq c$ (the power of the continuum). But it is well known that the power of the conjugate space of A is c so that Card $\hat{A} \leqq c$ and, hence, Card $E = c$. The proof is complete.

4. Remark. Suppose we are in case (b) of Theorem 1. Using the reasoning of [3, Corollary 4], one sees that there exists on E standard measures which are not canonical in the sense of [4].

5. Remark. Again, let A be a separable C^*-algebra. Let Prim (A) be the space of primitive ideals of A with the Jacobson topology. Let p be the canonical map $\pi \to \text{Ker}\,\pi$ from $\hat{A}$ onto Prim (A). The Jacobson topology on $\hat{A}$ is the inverse image by p of the Jacobson topology on Prim (A). We shall denote by Z_a (resp. Z_b) the set of all J in Prim (A) with the property a (resp. b) of Theorem 1. Thus, Prim A is the disjoint union of Z_a and Z_b, and $\hat{A}$ is the disjoint union of $Y_a = p^{-1}(Z_a)$ and $Y_b = p^{-1}(Z_b)$. From [6, p. 22, Theorem 2] and [1, 6.9.7], one sees that Y_a, Y_b, Z_a, Z_b are Borel for the Jacobson topologies, and that Y_a, Z_a are isomorphic standard Borel spaces. In order that a subset Y of $\hat{A}$ be Borel for the Jacobson topology, it is necessary and sufficient that Y be Borel for the Mackey structure, and that $Y = p^{-1}(p(Y))$. From [6, p. 70, Proposition 5], we know that both Y_a and Y_b may be dense in $\hat{A}$.

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