PROJECTIONS TO AUTOMORPHIC FUNCTIONS[1]

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1. **The main results.** Let $\bar{R}$ be a compact bordered Riemann surface with interior R. We represent R as the orbit space Δ/G where G is a finitely generated Fuchsian group of the second kind acting on the unit disk Δ. Choose a fundamental polygon $\mathcal{R}$ for G in Δ whose closure $\bar{\mathcal{R}}$ in the plane meets the boundary of Δ in a finite number of arcs, each of which corresponds to a boundary contour of R.

On the set of analytic functions in Δ we will consider the norms

$$\|f\|_\infty = \sup\{\,|f(z)| : z \in \Delta\}, \tag{1}$$

$$\|f\| = \iint_\Delta |f(z)|\,dxdy, \tag{2}$$

and the corresponding Banach spaces

$$H^\infty(\Delta) = \{f : \|f\|_\infty < \infty\} \quad \text{and} \quad A(\Delta) = \{\,f: \|f\| < \infty\}.$$

We shall also consider the subspaces $H^\infty(G) \subset H^\infty(\Delta)$ and $A(G) \subset A(\Delta)$ of functions which satisfy

$$f(Az) = f(z) \quad \text{for all } A \in G \quad \text{and} \quad z \in \Delta. \tag{3}$$

If f satisfies (3), then

$$\iint_\Delta |f(z)|\,dxdy = \iint_{\mathcal{R}} |f(z)| \Big(\sum_{A \in G} |A'(z)|^2 \Big) dxdy \tag{4}$$

so that $A(G)$ consists of those analytic functions which satisfy (3) and are summable over $\mathcal{R}$ with respect to the measure

$$dm(z) = \sum_{A \in G} |A'(z)|^2 dxdy. \tag{5}$$

Thus $H^\infty(G)$ corresponds to the space of bounded analytic functions on R, and $A(G)$ to the space of analytic functions on R which are summable with respect to dm.

THEOREM. *There is a projection P, bounded in the respective norms, which sends $A(\Delta)$ (resp. $H^\infty(\Delta)$) onto $A(G)$ (resp. $H^\infty(G)$) and has the following property: if $f(\zeta) = f(A\zeta)$ for some $\zeta \in \Delta$ and all $A \in G$, $f \in A(\Delta)$, then for any $g \in A(\Delta)$*

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(6) $$(Pfg)(\zeta) = f(\zeta)Pg(\zeta).$$

In particular, if $f \in A(G)$,

(7) $$P(fg) = fPg.$$

This Theorem is a direct consequence of the following.

LEMMA. *There is a polynomial* $F(z)$ *such that the Poincaré series*

(8) $$\Theta F(z) = \sum_{A \in G} F(Az)A'(z)^2$$

is bounded away from zero in the fundamental polygon $\mathcal{R}$.

A somewhat less general form of the Theorem is due to Forelli [6] who obtained a bounded projection P of $H^\infty(\Delta)$ onto $H^\infty(G)$ having property (7).

2. **Proof of the Lemma.** The set of limit points of G is a closed subset of the unit circle of linear measure zero. If Ω is the complement of the set of limit points in the extended plane, then Ω/G is a compact Riemann surface, the double of R. Let $R^* \supset \overline{R}$ be a subsurface of Ω/G such that (i) R^* is bounded by analytic curves, (ii) each component of $R^* - \overline{R}$ is a topological annulus, and (iii) $\pi(\infty)$ is in the exterior of R^* where $\pi\colon \Omega \to \Omega/G$ is the natural map. Then $\pi^{-1}(R^*) = D^*$ contains Δ, is invariant under G, and is bounded by a Jordan curve which is the union of $\pi^{-1}(\partial R^*)$ and the set of limit points of G. Moreover, $\overline{\mathcal{R}}$ is a compact subset of D^*. Let $\mathcal{R}^*$ be a fundamental "polygon" for G in D^* ($\mathcal{R}^*$ can be obtained, for example, by mapping D^* onto Δ).

By Abel's theorem there exists a meromorphic differential ω on the compact surface Ω/G which is analytic and nonzero on the closure of R^*. The quadratic differential ω^2 can be lifted to D^* to determine an analytic $\phi(z)$ which is nonzero in D^* and satisfies

(9) $$\phi(Az)A'(z)^2 = \phi(z) \quad \text{for all } z \in D^*,\ A \in G.$$

Furthermore, since ω^2 is analytic in the closure of R^*,

(10) $$\iint_{\mathcal{R}^*} |\phi(z)|\, dxdy < \infty.$$

We now appeal to a recent theorem of Bers [3] concerning Poincaré series in D^*. Let $Q(G)$ denote the Banach space of all functions $\phi(z)$ analytic in D^* which satisfy (9) and (10), the norm being given by (10). Bers has proved [3, Theorem 2] that the Poincaré series (8) defines a continuous map of $A(D^*)$ *onto* $Q(G)$. (A short proof of this theorem can be found in [4].) Furthermore, since D^* is a Jordan region, a theorem of O. J. Farrell [5] implies that the polynomials are dense in $A(D^*)$.

Applying these results to our nonzero function $\phi(z)$ in $Q(G)$, we obtain a sequence $\{F_n\}$ of polynomials such that $\Theta F_n \to \phi$ in $Q(G)$. But convergence in $Q(G)$ implies uniform convergence on compact sets in D^*. In particular, $\Theta F_n \to \phi$ uniformly in $\mathcal{R}$, and for sufficiently large n, ΘF_n is bounded away from zero in $\mathcal{R}$. q.e.d.

3. **Proof of the Theorem.** Choose a polynomial $F(z)$ in accordance with the Lemma. For $f \in A(\Delta)$ define

$$Pf(z) = (\Theta Ff)(z)/\Theta F(z). \tag{11}$$

Set

$$\delta^{-1} = \inf\{\,|\Theta F(z)| : z \in \mathcal{R}\} > 0,$$

$$M = \sup\left\{\sum_{A \in G} |A'(z)|^2 : z \in \mathcal{R}\right\} < \infty.$$

Then

$$\begin{aligned} \iint_{\mathcal{R}} |\Theta Ff|\, dxdy &\leqq \|F\|_\infty \iint_{\mathcal{R}} \sum_{A\in G} |f(Az)|\,|A'(z)|^2 dxdy \\ &= \|F\|_\infty \sum_{A\in G} \iint_{A(\mathcal{R})} |f(z)|\, dxdy = \|F\|_\infty \|f\|. \end{aligned} \tag{12}$$

Hence the series ΘFf converges absolutely and uniformly on compact subsets of $\mathcal{R}$ and therefore on compact subsets of Δ. Furthermore by the Lemma, ΘF is nonzero on Δ. Consequently, Pf is analytic in Δ; Pf obviously satisfies (3) and (6). We also have from (4) and (12) that

$$\begin{aligned} \|Pf\| &= \iint_{\Delta} |Pf(z)|\, dxdy = \iint_{\mathcal{R}} |Pf(z)|\, dm(z) \\ &\leqq M \iint_{\mathcal{R}} |Pf(z)|\, dxdy \leqq M\delta \iint_{\mathcal{R}} |\Theta Ff|\, dxdy \\ &\leqq M\delta \|F\|_\infty \|f\|, \end{aligned} \tag{13}$$

and, if f is in $H^\infty(\Delta)$ as well,

$$\begin{aligned} \|Pf\|_\infty &= \sup\{\,|Pf(z)| : z \in \Delta\} \\ &= \sup\{\,|Pf(z)| : z \in \mathcal{R}\} \leqq M\delta \|F\|_\infty \|f\|_\infty. \end{aligned} \tag{14}$$

The remainder of the Theorem follows at once. q.e.d.

4. Applications. As Forelli has pointed out [6, Corollary 2], Carleson's solution of the corona problem for $H^\infty(\Delta)$ and the existence of a projection $P: H^\infty(\Delta)\to H^\infty(G)$ with property (7) yield a solution of the corona problem for the compact bordered surface R. We state this consequence of Theorem 1 as

COROLLARY 1. *Let $\mathfrak{M}(R)$ be the maximal ideal space of the algebra $H^\infty(G)$. Then R is dense in $\mathfrak{M}(R)$.*

Other proofs of Corollary 1 have been given in [1], [6], [7].

COROLLARY 2. *Let S be a set of points in Δ which is invariant under G and let $\zeta(z)$ be a complex valued function on S such that $\zeta(Az)=\zeta(z)$ for all $A\in G$ and $z\in S$. There exists $f\in A(G)$ (resp. $H^\infty(G)$) with $f(z)=\zeta(z)$ for all $z\in S$ if and only if there exists $g\in A(\Delta)$ (resp. $H^\infty(\Delta)$) with $g(z)=\zeta(z)$, all $z\in S$.*

Indeed, if $g(z)$ is given, then $f(z)=Pg(z)=\zeta(z)$ for all $z\in S$.

Corollary 2 strengthens Stout's theorem [7] that if S is a G-invariant interpolating set for $H^\infty(\Delta)$ it is also an interpolating set for $H^\infty(G)$,

Let $\mathfrak{A}(G)\subset H^\infty(G)$ be the subspace of functions continuous in $\overline{\mathfrak{R}}$, and $\mathfrak{A}_0(G)\subset\mathfrak{A}(G)$ the subset of functions analytic in $\overline{\mathfrak{R}}$.

COROLLARY 3. *$\mathfrak{A}_0(G)$ is dense in $\mathfrak{A}(G)$.*

PROOF. We use the notations of §3. If $f\in\mathfrak{A}(G)$, set $f_r(z)=f(rz)$ for $r<1$. Then $P(f_r)\in\mathfrak{A}_0(G)$. We claim that $P(f_r)\to P(f)=f$ as $r\to 1$. For the proof, choose an enumeration $\{A_i\}$ of the elements of G, and set $C=\max(\|f\|_\infty, M)$. Given $\epsilon>0$ find N such that

$$\sup\left\{\sum_{i=N+1}^{\infty} |A_i'(z)|^2 : z\in\mathfrak{R}\right\} < \epsilon/4C\delta\|F\|_\infty$$

and find r_0 such that for $r_0<r<1$

$$\sup\left\{|f(rz)-f(z)| : z\in\bigcup_{i=1}^{N} A_i(\mathfrak{R})\right\} < \epsilon/2C\delta\|F\|_\infty.$$

Then for $z\in\mathfrak{R}$ and $r_0<r<1$,

$$|P(f-f_r)(z)| \leqq \|F\|_\infty\delta\sum_{i=1}^{N}|f(A_iz)-f_r(A_iz)|\,|A_i'(z)|^2 + 2C\|F\|_\infty\delta\sum_{i=N+1}^{\infty}|A_i'(z)|^2<\epsilon.$$

For another proof of Corollary 3, see [2, p. 291].

5. **Remarks.** A closer investigation of Forelli's projection of $H^\infty(\Delta)$ onto $H^\infty(G)$ shows that it too has property (6) but does not extend to $A(\Delta)$. Our projection P, like Forelli's projection, has the property that if $f \in L^p$ on $\{|z|=1\}$, $p \geqq 1$ then so is Pf. In the present case this fact is a simple consequence of the Hölder inequality and the convergence of $\Sigma|A'(z)|$ for $z \in \overline{\mathfrak{R}}$. In addition, P obviously has the following property. If $A_p(\Delta)$, $p \geqq 1$, is the Banach space of analytic functions in Δ with norm $\iint_\Delta |f(z)|^p dx\, dy < \infty$ and $A_p(G)$ is the subspace of functions satisfying (3), then P is a bounded projection of each $A_p(\Delta)$ onto $A_p(G)$.

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