1. The main results. Let \( \mathcal{R} \) be a compact bordered Riemann surface with interior \( R \). We represent \( R \) as the orbit space \( \Delta/G \) where \( G \) is a finitely generated Fuchsian group of the second kind acting on the unit disk \( \Delta \). Choose a fundamental polygon \( \mathfrak{R} \) for \( G \) in \( \Delta \) whose closure \( \overline{\mathfrak{R}} \) in the plane meets the boundary of \( \Delta \) in a finite number of arcs, each of which corresponds to a boundary contour of \( R \).

On the set of analytic functions in \( \Delta \) we will consider the norms

\[
(1) \quad \|f\|_\infty = \sup\{ |f(z)| : z \in \Delta \},
\]
\[
(2) \quad \|f\| = \iint_\Delta |f(z)| \, dx \, dy,
\]

and the corresponding Banach spaces

\[
H^\infty(\Delta) = \{ f : \|f\|_\infty < \infty \} \quad \text{and} \quad A(\Delta) = \{ f : \|f\| < \infty \}.
\]

We shall also consider the subspaces \( H^\infty(G) \subset H^\infty(\Delta) \) and \( A(G) \subset A(\Delta) \) of functions which satisfy

\[
(3) \quad f(Az) = f(z) \quad \text{for all} \ A \in G \quad \text{and} \quad z \in \Delta.
\]

If \( f \) satisfies (3), then

\[
(4) \quad \iint_\Delta |f(z)| \, dx \, dy = \iint_\mathfrak{R} |f(z)| \left( \sum_{A \in G} |A'(z)|^2 \right) \, dx \, dy
\]

so that \( A(G) \) consists of those analytic functions which satisfy (3) and are summable over \( \mathfrak{R} \) with respect to the measure

\[
(5) \quad dm(z) = \sum_{A \in G} |A'(z)|^2 \, dx \, dy.
\]

Thus \( H^\infty(G) \) corresponds to the space of bounded analytic functions on \( R \), and \( A(G) \) to the space of analytic functions on \( R \) which are summable with respect to \( dm \).

Theorem. There is a projection \( P \), bounded in the respective norms, which sends \( A(\Delta) \) (resp. \( H^\infty(\Delta) \)) onto \( A(G) \) (resp. \( H^\infty(G) \)) and has the following property: if \( f(\xi) = f(A\xi) \) for some \( \xi \in \Delta \) and all \( A \in G, f \in A(\Delta) \), then for any \( g \in A(\Delta) \)

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In particular, if $f \in A(G)$,

(7) \[ P(fg) = fPg. \]

This Theorem is a direct consequence of the following.

**Lemma.** There is a polynomial $F(z)$ such that the Poincaré series

(8) \[ \Theta F(z) = \sum_{A \in G} F(Az)A'(z)^2 \]

is bounded away from zero in the fundamental polygon $R$.

A somewhat less general form of the Theorem is due to Forelli [6] who obtained a bounded projection $P$ of $H^\omega(\Delta)$ onto $H^\omega(G)$ having property (7).

2. **Proof of the Lemma.** The set of limit points of $G$ is a closed subset of the unit circle of linear measure zero. If $\Omega$ is the complement of the set of limit points in the extended plane, then $\Omega/G$ is a compact Riemann surface, the double of $R$. Let $R^* \supseteq \overline{R}$ be a subsurface of $\Omega/G$ such that (i) $R^*$ is bounded by analytic curves, (ii) each component of $R^* - \overline{R}$ is a topological annulus, and (iii) $\pi(\infty)$ is in the exterior of $R^*$ where $\pi: \Omega \to \Omega/G$ is the natural map. Then $\pi^{-1}(R^*) = D^*$ contains $\Delta$, is invariant under $G$, and is bounded by a Jordan curve which is the union of $\pi^{-1}(\partial R^*)$ and the set of limit points of $G$. Moreover, $\overline{R}$ is a compact subset of $D^*$. Let $R^*$ be a fundamental “polygon” for $G$ in $D^*$ ($R^*$ can be obtained, for example, by mapping $D^*$ onto $\Delta$).

By Abel’s theorem there exists a meromorphic differential $\omega$ on the compact surface $\Omega/G$ which is analytic and nonzero on the closure of $R^*$. The quadratic differential $\omega^2$ can be lifted to $D^*$ to determine an analytic $\phi(z)$ which is nonzero in $D^*$ and satisfies

(9) \[ \phi(Az)A'(z)^2 = \phi(z) \quad \text{for all } z \in D^*, \ A \in G. \]

Furthermore, since $\omega^2$ is analytic in the closure of $R^*$,

(10) \[ \int \int_{R^*} |\phi(z)| \, dx \, dy < \infty. \]

We now appeal to a recent theorem of Bers [3] concerning Poincaré series in $D^*$. Let $Q(G)$ denote the Banach space of all functions $\phi(z)$ analytic in $D^*$ which satisfy (9) and (10), the norm being given by (10). Bers has proved [3, Theorem 2] that the Poincaré series (8) defines a continuous map of $A(D^*)$ onto $Q(G)$. (A short proof of this
Theorem can be found in [4]). Furthermore, since \( D^\ast \) is a Jordan region, a theorem of O. J. Farrell [5] implies that the polynomials are dense in \( A(D^\ast) \).

Applying these results to our nonzero function \( \phi(z) \) in \( Q(G) \), we obtain a sequence \( \{ F_n \} \) of polynomials such that \( \Theta F_n \rightarrow \phi \) in \( Q(G) \). But convergence in \( Q(G) \) implies uniform convergence on compact sets in \( D^\ast \). In particular, \( \Theta F_n \rightarrow \phi \) uniformly in \( \mathfrak{R} \), and for sufficiently large \( n \), \( \Theta F_n \) is bounded away from zero in \( \mathfrak{R} \). q.e.d.

3. Proof of the Theorem. Choose a polynomial \( F(z) \) in accordance with the Lemma. For \( f \in A(\Delta) \) define

\[
(11) \quad P_f(z) = (\Theta F_f)(z)/\Theta F(z).
\]

Set

\[
\delta^{-1} = \inf\{ |\Theta F(z)| : z \in \mathfrak{R} \} > 0,
\]

\[
M = \sup \Bigl\{ \sum_{A \in G} |A'(z)|^2 : z \in \mathfrak{R} \Bigr\} < \infty.
\]

Then

\[
\int \int_{\mathfrak{R}} |\Theta F_f| \, dx \, dy \leq \|F\|_\infty \int \int_{\mathfrak{R}} \sum_{A \in G} |f(Az)| \, |A'(z)|^2 \, dx \, dy
\]

\[
= \|F\|_\infty \sum_{A \in G} \int \int_{A(\mathfrak{R})} |f(z)| \, dx \, dy = \|F\|_\infty \|f\|.
\]

Hence the series \( \Theta F_f \) converges absolutely and uniformly on compact subsets of \( \mathfrak{R} \) and therefore on compact subsets of \( \Delta \). Furthermore by the Lemma, \( \Theta F \) is nonzero on \( \Delta \). Consequently, \( P_f \) is analytic in \( \Delta \); \( P_f \) obviously satisfies (3) and (6). We also have from (4) and (12) that

\[
\|P_f\| = \int \int_{\Delta} |P_f(z)| \, dx \, dy = \int \int_{\mathfrak{R}} |P_f(z)| \, dm(z)
\]

\[
\leq M \int \int_{\mathfrak{R}} |P_f(z)| \, dx \, dy \leq M\delta \int \int_{\mathfrak{R}} |\Theta F_f| \, dx \, dy
\]

\[
\leq M\delta \|F\|_\infty \|f\|,
\]

and, if \( f \) is in \( H^\infty(\Delta) \) as well,

\[
\|P_f\|_\infty = \sup \{ |P_f(z)| : z \in \Delta \}
\]

\[
= \sup \{ |P_f(z)| : z \in \mathfrak{R} \} \leq M\delta \|F\|_\infty \|f\|_\infty.
\]

The remainder of the Theorem follows at once. q.e.d.
4. Applications. As Forelli has pointed out [6, Corollary 2], Carleson’s solution of the corona problem for $H^\infty(\Delta)$ and the existence of a projection $P: H^\infty(\Delta) \to H^\infty(G)$ with property (7) yield a solution of the corona problem for the compact bordered surface $R$. We state this consequence of Theorem 1 as

**Corollary 1.** Let $\mathfrak{m}(R)$ be the maximal ideal space of the algebra $H^\infty(G)$. Then $R$ is dense in $\mathfrak{m}(R)$.

Other proofs of Corollary 1 have been given in [1], [6], [7].

**Corollary 2.** Let $S$ be a set of points in $\Delta$ which is invariant under $G$ and let $\xi(z)$ be a complex valued function on $S$ such that $\xi(Az) = \xi(z)$ for all $A \in G$ and $z \in S$. There exists $f \in A(G)$ (resp. $H^\infty(G)$) with $f(z) = \xi(z)$ for all $z \in S$ if and only if there exists $g \in A(\Delta)$ (resp. $H^\infty(\Delta)$) with $g(z) = \xi(z)$, all $z \in S$.

Indeed, if $g(z)$ is given, then $f(z) = Pg(z) = \xi(z)$ for all $z \in S$.

Corollary 2 strengthens Stout’s theorem [7] that if $S$ is a $G$-invariant interpolating set for $H^\infty(\Delta)$ it is also an interpolating set for $H^\infty(G)$.

Let $\mathcal{A}(G) \subset H^\infty(G)$ be the subspace of functions continuous in $\mathfrak{R}$, and $\mathcal{A}_0(G) \subset \mathcal{A}(G)$ the subset of functions analytic in $\mathfrak{R}$.

**Corollary 3.** $\mathcal{A}_0(G)$ is dense in $\mathcal{A}(G)$.

**Proof.** We use the notations of §3. If $f \in \mathcal{A}(G)$, set $f_r(z) = f(rz)$ for $r < 1$. Then $P(f_r) \in \mathcal{A}_0(G)$. We claim that $P(f_r) \to P(f)$ as $r \to 1$. For the proof, choose an enumeration $\{A_i\}$ of the elements of $G$, and set $C = \max(||f||_\infty, M)$. Given $\varepsilon > 0$ find $N$ such that

$$\sup \left\{ \sum_{i=N+1}^{\infty} |A_i(z)|^2 : z \in \mathfrak{R} \right\} < \varepsilon/4C\delta||P||_\infty$$

and find $r_0$ such that for $r_0 < r < 1$

$$\sup \left\{ |f_r(z) - f(z)| : z \in \bigcup_{i=1}^{N} A_i(\mathfrak{R}) \right\} < \varepsilon/2C\delta||P||_\infty.$$  

Then for $z \in \mathfrak{R}$ and $r_0 < r < 1$,

$$|P(f - f_r)(z)| \leq ||P||_\infty \delta \sum_{i=1}^{N} |f(A_i z) - f_r(A_i z)| \cdot |A_i(z)|^2$$

$$+ 2C||P||_\infty \delta \sum_{i=N+1}^{\infty} |A_i(z)|^2 < \varepsilon.$$  

For another proof of Corollary 3, see [2, p. 291].
5. Remarks. A closer investigation of Forelli’s projection of $H^\infty(\Delta)$ onto $H^\infty(G)$ shows that it too has property (6) but does not extend to $A(\Delta)$. Our projection $P$, like Forelli’s projection, has the property that if $f \in L^p$ on $\{|z| = 1\}$, $p \geq 1$, then so is $Pf$. In the present case this fact is a simple consequence of the Hölder inequality and the convergence of $\Sigma |A'(z)|$ for $z \in \mathbb{R}$. In addition, $P$ obviously has the following property. If $A_p(\Delta)$, $p \geq 1$, is the Banach space of analytic functions in $\Delta$ with norm $\int_\Delta |f(z)|^p \, dx \, dy < \infty$ and $A_p(G)$ is the subspace of functions satisfying (3), then $P$ is a bounded projection of each $A_p(\Delta)$ onto $A_p(G)$.

References


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