1. Introduction. Let \((\Omega, \Sigma, \lambda)\) be a measure space with \(\lambda(\Omega) = 1\). Consider the set \(\mathcal{A}\) of all the operators, \(P\), on \(L_2(\Omega, \Sigma, \lambda)\) such that:

1.1) \(\|P\| = 1\)

1.2) \(P1 = 1\)

1.3) If \(f \in L_2\) and \(f \geq 0\) then \(Pf \geq 0\) a.e.

These operators are given by a Markov transition function for which \(\lambda\) is an invariant measure.

The set \(\mathcal{A}\) is a weakly closed convex set and is selfadjoint.

We shall study several notions that are related to “mixing” properties and will show:

- a convex combination mixes better than its generators.

Many of the results can be phrased, and their proofs are identical, for general contractions on a Hilbert space. This will be mentioned without details to avoid repetitions.

Let us repeat the following definitions from [1]. For each \(P \in \mathcal{A}\):

1.4) \(K(P) = \{f : f \in L_2(\lambda), \|Pn f\| = \|P^* n f\| = \|f\|, n = 1, 2, \cdots \}\).

1.5) \(H_0(P) = \{f : f \in L_2(\lambda), \text{ weak lim } P^* n f = 0\}\).

1.6) \(H_1(P) = H_0(P)\).

It is proved in [1, Theorems 2 and 4] that these are subspaces and \(H_1(P) \subseteq K(P)\) and

1.7) \(K(P) = L_2(\Omega_1, \Sigma_1, \lambda) \Sigma_1 \subseteq \Sigma\) and \(\cup \Sigma_1 = \Omega_1\).

Except for (1.7) all these definitions and results are valid for general contractions on a Hilbert space.

Our aim will be to find how “small” is \(H_1(P)\). Thus if \(\Sigma_1\) is atomic the “Limit Theorem” [1, Theorem 8] holds and if \(H_1 = \text{constant functions}\), then \(P\) is strongly mixing.

The notions of ergodic and strongly mixing operators are defined in [3].

2. Convex combination of finitely many Markov transition functions.

**Theorem 1.** Let \(P_1, \cdots, P_m\) belong to \(\mathcal{A}\) and \(P = \Sigma \alpha_i P_i\) where \(\alpha_i > 0\) and \(\Sigma \alpha_i = 1\). Then
\[ K(P) = \bigcap_{i=1}^{m} K(P_i) \cap \bigcap_{n=1}^{\infty} \{ f: P_i^n f = P_i^n f, P_i^* f = P_i^* f \text{ for all } 1 \leq i, j \leq m \}. \]

**Proof.** It is enough to consider the case \( m = 2 \) since a convex combination of \( P_1, \ldots, P_m \) can be represented as a convex combination of \( P_1 \) and \( Q \) where \( Q \) is a convex combination of \( P_2, \ldots, P_m \).

Let \( f \in K(P) \) then
\[
\|f\| = \|\alpha P_1 + (1 - \alpha) P_2 \| f\| \leq \sum_{j} (1 - \alpha)^{n-j} \| P_{1}^{h_{1}} P_{2}^{k_{1}} \cdots P_{1}^{h_{n}} P_{2}^{k_{n}} f\|
\]
\[
\leq \sum_{j} (1 - \alpha)^{n-j} \|f\| = \|f\|
\]
where \( h_{i}, k_{i} \) assume the values 0, 1 and \( \sum h_{i} = j, \sum k_{i} = n - j \) and the sum is taken over all such choices when \( j \) ranges between 0 and \( n \). Thus each of the terms in the sum must have the norm \( \|f\| \) (since the operators \( P_1 \) and \( P_2 \) are contractions). In particular \( \|P_{1}^{h_{1}} P_{2}^{k_{1}} \cdots P_{1}^{h_{n}} P_{2}^{k_{n}} f\| = \|f\| \). Also a Hilbert space is strictly convex and equality can occur in the triangular inequality only when the terms are proportional. Thus \( P_{1}^{n} f = \gamma P_{2}^{n} f \) for some \( \gamma \geq 0 \) and since they have the same norms \( P_{1}^{n} f = P_{2}^{n} f \). Finally if \( f \in K(P) \) then \( \|P_{1}^{*n} f\| = \|f\| \) and the same argument would apply to \( P_{1}^{*} \) and \( P_{2}^{*} \).

Conversely let \( P_{1}^{n} f = P_{2}^{n} f, P_{1}^{*n} f = P_{2}^{*n} f \) and \( f \in K(P_1) \). Then
\[
P_{1}^{h_{1}} P_{2}^{k_{1}} \cdots P_{1}^{h_{n}} P_{2}^{k_{n}} f = P_{1}^{h_{1}} P_{2}^{k_{1}} \cdots P_{1}^{h_{n}} P_{1}^{*n} f = P_{1}^{h_{1}} P_{2}^{k_{1}} \cdots P_{2}^{*n} f
\]
continuing in this way we shall get \( P_{1}^{n} f \) and thus \( P_{1}^{n} f = P_{2}^{n} f \). Now since \( f \in K(P_1) \), \( \|P^{n} f\| = \|P_{1}^{n} f\| = \|f\| \). The result for \( P_{1}^{*} \) is analogous.

**Theorem 2.** Let \( P_1 \cdots P_m \in \mathfrak{S} \) and \( P = \Sigma \alpha_i P_i \), where \( \alpha_i > 0 \) and \( \Sigma \alpha_i = 1 \). Then
\[ H_{1}(P) = \bigcap_{i=1}^{m} H_{1}(P_i) \cap \bigcap_{n=1}^{\infty} \{ f: P_{i}^{n} f = P_{i}^{n} f, P_{i}^{*n} f = P_{i}^{*n} f \text{ for all } 1 \leq i, j \leq m \}. \]

**Proof.** If \( f \in H_{1}(P) \) then \( f \in K(P) \) hence \( P_{n}^{*} f = P_{i}^{*n} f \) for every \( 1 \leq i \leq m \). By [2, Theorem 3.1] \( f \in H_{1}(P_i) \). The converse is proved in the same way.

Note that Theorems 1 and 2 hold for any contraction in a Hilbert space.

From (2.1) follows that, under the assumptions of Theorem 1, \( \Sigma_1 (P) \subseteq \bigcap_{i=1}^{m} \Sigma_1 (P_i) \). Hence if at least one of the fields \( \Sigma_1 (P_i) \) is atomic, then so is \( \Sigma_1 (P) \).
From (2.2) follows, under the assumptions of Theorem 2, that if at least one of the operators \( P_i \) is strongly mixing so is \( P \). Thus if \( P_1 \) is strongly mixing and \( P_2 \) any operator in \( \mathcal{A} \) then \( \alpha P_1 + (1 - \alpha) P_2 \) is strongly mixing for any choice of \( 0 < \alpha < 1 \). Thus \( P_2 \) can be approximated in norm by strongly mixing operators. Let \( Q \) be an invertible ergodic transformation in \( \mathcal{A} \). By the Second Category Theorem [3, p. 78] such transformations exist. Put \( P = \frac{1}{2} (Q + Q^2) \). Now if \( f \in K(P) \) then \( Qf = Q^2 f \) by Theorem 1 hence \( f = Qf \) and \( f \) is a constant. This shows that there is at least one strongly mixing operator in \( \mathcal{A} \) and hence a dense set of \( \mathcal{A} \). Notice that we did not show the existence of strongly mixing transformations but only operators in \( \mathcal{A} \).

Let us conclude this chapter with the following remark: Let \( P \) and \( P_1 \) belong to \( \mathcal{A} \). There exists an operator \( P_2 \) in \( \mathcal{A} \) such that \( P \) is a convex combination of \( P_1 \) and \( P_2 \) iff for some \( 0 < \alpha < 1 \) \( P f \geq \alpha P_1 f \) for every \( f \geq 0 \).

Clearly the condition is necessary. Now if \( P f \geq \alpha P_1 f \) for every \( f \geq 0 \) define \( P_2 f = (1 - \alpha)^{-1} (P f - P_1 f) \). Then \( P_2 \) satisfies (1.2) and (1.3). In order to prove (1.1) it is enough to observe that \( P_2 \) is a contraction on \( L^\infty \) and if \( f = \sum c_i 1_{A_i} \), where \( A_i \) are disjoint sets and \( 1_{A_i} \) denote their characteristic functions, then

\[
\int |P_2 f| \, d\lambda \leq \sum |c_i| (1 - \alpha)^{-1} \int (P - \alpha P_1) 1_{A_i} d\lambda = \sum |c_i| \lambda(A_i)
\]

since

\[
\int (P - \alpha P_1) 1_{A_i} d\lambda = \langle (P - \alpha P_1) 1_{A_i}, 1 \rangle = \langle 1_{A_i}, (P^* - \alpha P_1^*) 1 \rangle = (1 - \alpha) \lambda(A_i),
\]

where \( \langle f, g \rangle \) is the inner product of \( f \) and \( g \).

3. Integral averages. Following Choquet's theory let us consider, throughout this chapter, the following setup:

(3.1) Let \( \mu \) be a regular positive measure, of total mass 1, defined on the Borel subsets of \( \mathcal{A} \) with its weak operator topology. Put

\[
Q = \int \mu (dP).
\]

The operator \( Q \) is defined by \( \langle Qf, g \rangle = f \langle Pf, g \rangle \mu (dP) \). The integral exists since for every pair of vectors \( f, g \) the function \( \phi(P) = \langle Pf, g \rangle \) is continuous in the weak operator topology. Thus \( \int \mu (Pf, g) \mu (dP) \) defines a bilinear form and hence is equal to \( \langle Qf, g \rangle \) for some operator \( Q \). It is easy to check that \( Q \) belongs to \( \mathcal{A} \).
Let us consider all the open subsets of $\mathfrak{A}$ on which $\mu$ vanishes. Since $\mu$ is regular $\mu$ vanishes also on the union of all these sets. Denote by $\mathfrak{B}$ (support of $\mu$) the complement of this set. Thus

\[ P_0 \in \mathfrak{B} \text{ iff } \mu \text{ does not vanish on any neighborhood of } P_0. \]

**Theorem 3.** Given $Q$ by (3.1) then

\[ K(Q) = \bigcap_{P \in \mathfrak{B}} K(P) \cap \bigcap_{P \in \mathfrak{B}} \{ f : P^nf = Q^nf, P^{*n}f = Q^{*n}f \text{ for all } n \}. \]

**Proof.** Let $f \in K(Q)$ and $P_0 \in \mathfrak{B}$ and let $\mathfrak{D}$ be any weak neighborhood of $P_0$. Put

\[ Q = \mu(\mathfrak{D})^{-1} \int_\mathfrak{D} P\mu(dP) + \mu(\mathfrak{B} - \mathfrak{D})^{-1} \int_{\mathfrak{B} - \mathfrak{D}} P\mu(dP). \]

Then, by Theorem 1, $(\mu(\mathfrak{D})^{-1} \int_\mathfrak{D} P\mu(dP))(f) = Qf$. Or, for any $g \in L_2, \mu(\mathfrak{D})^{-1} \int_\mathfrak{D} \langle Pf, g \rangle \mu(dP) = \langle Qf, g \rangle$. If for some $g \langle Pf, g \rangle \neq \langle Qf, g \rangle$ say $\langle Pf, g \rangle < \langle Qf, g \rangle$ then taking for $\mathfrak{D}$ the open set $\{ P : \langle Pf, g \rangle < \langle Qf, g \rangle \}$ we shall get a contradiction. Thus for every $P \in \mathfrak{B}, Pf = Qf$. Now $P^nf = P^{n-1}Qf$ and $Qf \in K(Q)$ too and by an induction argument $P^nf = Qf$. The argument for $P^*$ is analogous.

Conversely, if $f$ belongs to the right side of (3.3) then take any $P$ in $\mathfrak{B}$ and $\|f^*\|^2 = \|f\|^2$, since $f \in K(P)$.

**Remark.** Theorem 3 can be viewed as a necessary condition for an element $P$ of $\mathfrak{A}$ to belong to the support of any representation of the type (3.1).

**Theorem 4.** Given $Q$ by 3.1 then

\[ H_1(Q) = K(Q) \cap \bigcap_{P \in \mathfrak{B}} H_1(P). \]

The proof is identical to the proof of Theorem 2.

4. Semigroup of contractions. Let us conclude this note with a study of convergence of iterates of the resolvent of a semigroup of contractions. The situation is somewhat similar to the one studied in Chapter 3 but much stronger results are valid.

Let $P_t$ be a strongly continuous semigroup of contractions in the Hilbert space $H$. Let $R_\lambda = \int_0^\infty e^{-\lambda t} P_t dt, \lambda > 0$. Thus $R_\lambda, \lambda > 0$, is the resolvent of the infinitesimal generator, $A$, of $P_t$ at the point $\lambda$. Let $U_t$ be the strong dilation of the semigroup see [4, Theorem IV]. Then $U_t$ is a strongly continuous semigroup of unitary operators. Let the infinitesimal generator of $U_t$ be $iB$. Then $B$ is selfadjoint [5, p. 385]. Thus
\[
\int_0^\infty e^{-\lambda t} U_t dt = (\lambda - iB)^{-1} \quad \lambda > 0.
\]

The spectrum of \(\lambda(\lambda - iB)^{-1}\) is included in \(\{\lambda(\lambda - it)^{-1}: t \text{ is real}\}\). This set is inside the unit circle and touches the circumference of the unit circle at the point 1 only. Now \(\lambda(\lambda - iB)^{-1}\) is a normal operator and, from the Spectral Theorem and the above description of the spectrum, follows that \((\lambda(\lambda - iB)^{-1})^n f\) is strongly convergent for every \(f\) in the larger space where the \(U_t\) are defined.

**Theorem 5.** Let \(R_\lambda = \int_0^\infty e^{-\lambda t} P_t dt, \lambda > 0\). Then \(\lim (\lambda R_\lambda)^n f = \text{projection of } f \text{ on the set } \{g: P_t g = g \text{ for all } t > 0\}\).

**Proof.** Let \(L = \{g: P_t g = g \text{ for all } t\}\) and \(f = f_1 + f_2\) where \(f_1 \subseteq L\) and \(f_2 \perp L\). Clearly \(\lambda R_\lambda f_1 = f_1\) and we shall consider \(f_2\) only. Now \(\|P_t\| \leq 1\) and thus \(P_t g = g\) if and only if \(P_t^* g = g\) or \(L^\perp\) is invariant under \(P_t\). Now

\[
(\lambda R_\lambda)^n = \lambda^n \int_0^\infty \cdots \int_0^\infty e^{-\lambda(t_1 + \cdots + t_n)} P_{t_1 + \cdots + t_n} dt_1 \cdots dt_n
\]

and is equal to the projection of \([\lambda(\lambda - iB)^{-1}]^n\) on \(H\). Thus \((\lambda R_\lambda)^n f_2\) is strongly convergent too. Let its limit be \(h\). Then \(\lambda R_\lambda h = h\). Hence \(h\) belongs to the domain of definition of \(A\) and \((A - \lambda) h = \lambda(A - \lambda) R_\lambda h = \lambda h\) or \(Ah = 0\). Thus \(h \in L\) but \(f_2\) and \((\lambda R_\lambda)^n f_2\) belong to \(L\). Therefore \(h = 0\).

**References**


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