COINCIDING DIRECTIONS FOR IMBEDDINGS OF SPACES[1]

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W. T. Wu [4] and [5] and A. Shapiro [3] defined obstruction classes $w^i(X)$ to the existence of an imbedding $f: X \to R^n$ of a topological space X in the euclidean n-space R^n. The Wu-Shapiro classes w^i are particular cases of the Smith classes defined for a space with an involution. We recall briefly the definition.

1. **Equivariant cohomology.** Let A be a space with a continuous fixed point free involution $\alpha: A \to A$. Consider the two possible actions of α on the coefficient group $\boldsymbol{Z}$ of integers: the trivial action and the nontrivial one. Let $\overset{\circ}{H}{}^i_+(A)$ (resp. $\overline{H}^i_+(A)$) denote the equivariant (resp. residual) ith cohomology group of A for the trivial action; and let $\overset{\circ}{H}{}^i_-(A)$ (resp. $\overline{H}^i_-(A)$) denote the corresponding equivariant (resp. residual) cohomology groups for the nontrivial action of α on $\boldsymbol{Z}$ (the singular homology is considered throughout; see [1]). The groups $\overset{\circ}{H}{}^i_+(A)$ and $\overset{\circ}{H}{}^i_-(A)$ can also be described as the corresponding cohomology groups of the orbit space A/α, with the constant and twisted coefficients $\{\boldsymbol{Z}\}$, respectively.

There are exact sequences:

$$(1.1) \qquad \cdots \to H^{i-1}(A) \to \overline{H}^{i-1}_+(A) \xrightarrow{\delta_i} \overset{\circ}{H}{}^i_+(A) \to H^i(A) \to \cdots,$$

$$(1.2) \qquad \cdots \to H^{i-1}(A) \to \overline{H}^{i-1}_-(A) \to \overset{\circ}{H}{}^i_-(A) \to H^i(A) \to \cdots.$$

Consider the singular chain complex $C(A)$ of A and the chain maps $1+(-1)^i\alpha_i$, where $\alpha_i: C_i(A) \to C_i(A)$ is induced by α. They induce homomorphisms

$$t^i: \overline{H}^i_-(A) \to \overset{\circ}{H}{}^i_+(A) \quad \text{if } i \text{ is even},$$

$$t^i: \overline{H}^i_+(A) \to \overset{\circ}{H}{}^i_-(A) \quad \text{if } i \text{ is odd}.$$

The homomorphisms t^i are known to be isomorphisms (see [5]). Therefore we shall identify the groups $\overline{H}^i_-(A)$ and $\overset{\circ}{H}{}^i_+(A)$, if i is even, and the groups $\overline{H}^i_+(A)$ and $\overset{\circ}{H}{}^i_-(A)$, if i is odd, under the isomorphisms

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t^i and denote them by $\overset{\circ}{H}{}^i(A)$. Thus $\overset{\circ}{H}{}^i$ denotes $\overset{\circ}{H}{}^i_+ = \overline{H}^i_-$, if i is even, and $\overset{\circ}{H}{}^i_- = \overline{H}^i_+$, if i is odd.

Let $\eta^p: \overset{\circ}{H}{}^0(A) \to \overset{\circ}{H}{}^p(A)$ denote the homomorphism defined by $\eta^p = \delta^{p-1} \circ (t^{p-1})^{-1} \circ \cdots \circ \delta^0 \circ (t^0)^{-1}$.

In particular, $\overset{\circ}{H}{}^0(A) = \overset{\circ}{H}{}^0_+(A)$ and it contains the constant unit cohomology class $1 \in \overset{\circ}{H}{}^0(A)$. The cohomology class

$$s^p(A) = \eta^p(1) \in \overset{\circ}{H}{}^p(A)$$

is called the pth Smith class of A. The index Ind (A) of A is defined to be the maximal integer p such that $s^p(A) \neq 0$ (or ∞ if $s^p(A) \neq 0$ for all integers $p \geqq 0$).

If $A = S^n$ and $\alpha: S^n \to S^n$ is the antipodal involution, then $H^i(S^n) \cong \boldsymbol{Z}_2$, for $0 < i \leqq n$, generated by $s^i(S^n)$; and Ind $(S^n) = n$.

We use a similar notation for homology. Then we have dual isomorphisms

$$t_i: \overset{\circ}{H}{}^-_i(A) \to \overline{H}^+_i(A) \quad \text{if } i \text{ is even,}$$

$$t_i: \overset{\circ}{H}{}^+_i(A) \to \overline{H}^-_i(A) \quad \text{if } i \text{ is odd;}$$

they are induced by the chain map $1 + (-1)^{i-1}\alpha_i: C_i(A) \to C_i(A)$. We again identify the corresponding groups under t_i and denote them by $\overset{\circ}{H}_i(A)$. Thus $\overset{\circ}{H}_i$ denotes $\overset{\circ}{H}{}^-_i = \overline{H}^+_i$, if i is even, and $\overset{\circ}{H}{}^+_i = \overline{H}^-_i$, if i is odd.

It is easy to verify that the Kronecker index pairing $\langle\ ,\ \rangle: \overset{\circ}{H}_i(A) \otimes \overset{\circ}{H}{}^i(A) \to \boldsymbol{Z}_2$ is defined.

If A and B are two spaces with fixed point free involutious and $\phi: A \to B$ is an involution preserving map, then $\phi^* s^p(B) = s^p(A)$ and thus Ind $(A) \leqq$ Ind (B).

A space A is said to be (-1)-acyclic if it is nonempty; it is said to be p-acyclic, $p \geqq 0$, if its cohomology groups $H^i(A)$ are isomorphic to those of a one-point space in dimensions $i \leqq p$.

Observe that if A is p-acyclic, then the connecting homomorphisms δ^{i-1} in sequences (1.1) and (1.2) are isomorphisms, for $i \leqq p$, and monomorphisms for $i = p+1$. Thus if A is p-acyclic, then Ind $(A) \geqq p+1$.

2. **Case** mod 2. We obtain a simplified version of the above theory if we use the group $\boldsymbol{Z}_2$ as coefficient group. In this case there is only one action of α on the coefficient group, the trivial one, and sequences (1.1) and (1.2) coincide. We denote the resulting homomorphisms analogous to η^p by

$$\overset{p}{\eta_2}: \overset{\circ}{H}{}^0(A; \boldsymbol{Z}_2) \to \overset{\circ}{H}{}^p(A; \boldsymbol{Z}_2);$$

the Smith class mod 2 by $s_2^p(A) = \eta_2^p(1)$; and the index mod 2 by $\mathrm{Ind}_2(A)$.

Observe that we have the following commutative diagram (for a nonempty space A):

$$\begin{array}{ccc} \overset{\circ}{H}{}^0(A) & \xrightarrow{\eta^p} & \overset{\circ}{H}{}^p(A) \\ \downarrow & & \downarrow \\ \overset{\circ}{H}{}^0(A; \boldsymbol{Z}_2) & \xrightarrow{\overset{p}{\eta_2}} & \overset{\circ}{H}{}^p(A; \boldsymbol{Z}_2). \end{array}$$

Here the vertical arrows denote the reduction of the coefficients mod 2. Since the homomorphism $\overset{\circ}{H}{}^0(A) \to \overset{\circ}{H}{}^0(A; \boldsymbol{Z}_2)$ carries the unit cohomology class of $\overset{\circ}{H}{}^0(A)$ to the unit cohomology class of $\overset{\circ}{H}{}^0(A; \boldsymbol{Z}_2)$, it follows that

$$\mathrm{Ind}_2(A) \leqq \mathrm{Ind}(A). \tag{2.1}$$

REMARK (2.2). It is easily seen that $\mathrm{Ind}_2(A)$ coincides with the index of A as defined in [6]. Also, $\mathrm{Ind}_2(A) \geqq p$ if and only if A contains a (p, α)-system in the sense of [2].

3. **Imbeddings into R^n and Wu-Shapiro classes.** Let X be a space and let $\tilde{X}^*$ denote the deleted square $X \times X - \Delta$, where Δ is the diagonal. Let $\alpha: \tilde{X}^* \to \tilde{X}^*$ be the involution permuting the coordinates $\alpha(x_1, x_2) = (x_2, x_1)$. Then $w^p(X) = s^p(\tilde{X}^*) \in \overset{\circ}{H}{}^p(\tilde{X}^*)$ are the Wu-Shapiro classes of X. We also denote

$$I(X) = \mathrm{Ind}(\tilde{X}^*), \qquad I_2(X) = \mathrm{Ind}_2(\tilde{X}^*).$$

If $f: X \to R^n$ is a topological imbedding, then the map $\phi: \tilde{X}^* \to S^{n-1}$ defined by

$$\phi(x_1, x_2) = (f(x_1) - f(x_2)) / \left| f(x_1) - f(x_2) \right|$$

is involution preserving (for the antipodal involution on S^{n-1}). Since S^{n-1} is a deformation retract of $\tilde{S}^{n-1*}$ by a deformation which preserves the involution α, it follows that $I(X) < n$. Thus we have

THEOREM (3.1) (WU-SHAPIRO). *If X is imbeddable in R^n, then $I(X) < n$, i.e. $w^n(X) = 0$.*

This means that $w^n(X)$ is an obstruction to the existence of an imbedding $f: X \to R^n$. Wu [5] and Shapiro [3] proved that if X is a finite polyhedron and $n > 2$, then the condition $w^{2n}(X) = 0$ is also sufficient for the imbeddability of X in R^{2n}.

Suppose now that $f: X \to R^n$ and $g: Y \to R^n$ are (topological) imbeddings of spaces X and Y in R^n. We say that f and g have a coinciding direction if there exist two pairs of distinct points $x_1, x_2 \in X$, $x_1 \neq x_2$; $y_1, y_2 \in Y$, $y_1 \neq y_2$, such that the lines

$$\overline{f(x_1)f(x_2)} \quad \text{and} \quad \overline{g(x_1)g(x_2)}$$

are parallel.

The purpose of this paper is to prove the following

THEOREM (3.2). *If $I(X)+I_2(Y) \geqq n$, then any imbeddings $f: X \to R^{n+1}$, $g: Y \to R^{n+1}$ have a coinciding direction.*

Theorem (3.1) can be obtained as a special case of (3.2): if there exists an imbedding $f: X \to R^n \subset R^{n+1}$, let Y be a space consisting of two points, and let $g: Y \to R^{n+1}$ be an imbedding of Y into a line in R^{n+1} orthogonal to R^n. Then f and g have no coinciding direction and thus $I(X)+I_2(Y)=I(X)<n$.

Actually, a somewhat stronger result than (3.2) will be proved; but the question whether the condition $I(X)+I(Y) \geqq n$ is also sufficient for the existence of a coinciding direction for f and g remains open.

EXAMPLE (3.3). For $X=Y=S^1$, we have $I(S^1)=I_2(S^1)=1$. Thus any two simple closed curves in R^3 have a pair of parallel chords. Similarly, if Y is a triod (a space homeomorphic to the letter Y), then $I(Y)=I_2(Y)=1$. Thus any two triods or a triod and a simple closed curve in R^3 have coinciding directions.

4. **Involution preserving maps into S^n and the intersection homomorphism.** Let A and B be spaces with involutions $\alpha: A \to A$, $\beta: B \to B$, and let $\phi: A \to S^n$, $\psi: B \to S^n$ be involution preserving maps (for the antipodal involution of S^n). Let p and q be integers such that $p+q=n$. Then ϕ, ψ define a natural intersection homomorphism

$$\Omega(\phi, \psi): \mathring{H}_q(B) \to \mathring{H}^p(A).$$

The definition of $\Omega(\phi, \psi)$ can be sketched as follows. Let T be an antipodal cell decomposition of S^n. For each homology class $\xi_q \in \mathring{H}_q(B)$, the class $\psi_*(\xi_q)$ may be represented by an (equivariant) cycle z_q of T. The cycle z_q defines a homomorphism $u^p: C_p(A) \to \boldsymbol{Z}$ as follows: for each chain $c_p \in C_p(A)$ we can place z_q and c_p in general position and define $u^p(c_p)$ to be the intersection number of z_q and c_p. Since z_q is equivariant, the cochain u^p is also equivariant and it is actually a cocycle. We define $\Omega(\phi, \psi)\xi_q$ to be the cohomology class represented by u^p.

By considering two dual antipodal cell decompositions T and T^* of S^n, it can be seen that the (unique) isomorphism $\mathring{H}_q(S^n) \cong \mathring{H}^p(S^n)$ $(\cong \boldsymbol{Z}_2)$ is just the homomorphism

$$\omega = \Omega(1_{S^n}, 1_{S^n}) : \mathring{H}_q(S^n) \to \mathring{H}^p(S^n)$$

which we denote briefly by ω; 1_{S^n} is the identity map of S^n; thus ω is the Poincaré isomorphism. It follows that the intersection homomorphism $\Omega(\phi, \psi)$ can be defined equivalently by the following commutative diagram

$$\begin{array}{ccc} \mathring{H}_q(B) & \xrightarrow{\Omega(\phi,\psi)} & \mathring{H}^p(A) \\ \psi_* \Big\downarrow & & \Big\uparrow \phi^* \\ \mathring{H}_q(S^n) & \xrightarrow{\omega} & \mathring{H}^p(S^n), \end{array}$$

i.e. $\Omega(\phi, \psi) = \phi^* \omega \psi_*$.

PROPOSITION (4.1). *If A and B are spaces with involutions and $\phi: A \to S^n$, $\psi: B \to S^n$ are involution preserving maps such that $\phi(A) \cap \psi(B) = \varnothing$, then $\Omega(\phi, \psi) = 0$.*

For $\Omega(\phi, \psi)$ is defined by means of intersection numbers.

PROPOSITION (4.2). *If ϕ, ψ are as in Proposition* (4.1) *and $\xi_q \in \mathring{H}_q(B)$ is a homology class such that the Kronecker index $\langle \xi_q, s^q(B) \rangle \neq 0$, then $\Omega(\phi, \psi)\xi_q = s^p(A)$.*

PROOF. By the naturality of the Kronecker index and the Smith classes, it follows that $\langle \psi_*(\xi_q), s^q(S^n) \rangle \neq 0$; but this means that $\psi_*(\xi_q)$ is the unique nonzero element of $\mathring{H}_q(S^n)$. Consequently, $\omega\psi_*(\xi_q)$ is the unique nonzero element of $\mathring{H}^p(S^n)$, i.e. the Smith class $\omega\psi_*(\xi_q) = s^p(S^n)$. It follows that $\Omega(\phi, \psi) = \phi^* \omega \psi_*(\xi_q) = s^p(A)$.

Due to the fact that there is a unique isomorphism $\iota: \mathring{H}_q(S^n; \boldsymbol{Z}_2) \cong H_q(S^n)$ $(\cong \boldsymbol{Z}_2)$, we can define an intersection homomorphism

$$\bar{\Omega}(\phi, \psi) = \phi^* \omega \iota \psi_* : \mathring{H}_q(B; \boldsymbol{Z}_2) \to \mathring{H}^p(A).$$

Propositions corresponding to Propositions (4.1) and (4.2) hold true:

PROPOSITION (4.3). *Under the assumptions of Proposition* (4.1), $\Omega(\phi, \psi) = 0$.

PROPOSITION (4.4). *If $\xi_q \in \overset{\circ}{H}_q(B; \boldsymbol{Z}_2)$ is a homology class such that the Kronecker index $\langle \xi_q, s^q(B)\rangle \neq 0$, then $\bar{\Omega}(\phi, \psi)\xi_q = s^p(A)$.*

For we have again $\langle \psi_*(\xi_q), s_2^q(S^n)\rangle \neq 0$; it follows that $\psi_*(\xi_q)$ is the nonzero element of $\overset{\circ}{H}_q(S^n; \boldsymbol{Z}_2)$, hence $\iota\psi_*$ is the nonzero element of $\overset{\circ}{H}_q(S^n)$ and $\omega\iota\psi_*(\xi_q) = s^p(S^n)$. Thus $\bar{\Omega}(\phi, \psi)\xi_q = s^p(A)$.

THEOREM (4.5). *If A and B are spaces with involutions such that $\mathrm{Ind}(A) + \mathrm{Ind}_2(B) \geqq n$ and $\phi: A \to S^n, \psi: B \to S^n$ are involution preserving maps, then $\phi(A) \cap \psi(B) \neq \varnothing$.*

PROOF. There exist integers p, q with $p+q=n$ such that $s^p(A) \neq 0$, $s_2^q(B) \neq 0$. Since $\overset{\circ}{H}{}^q(B; \boldsymbol{Z}_2) = \mathrm{Hom}(\overset{\circ}{H}_q(B; \boldsymbol{Z}_2), \boldsymbol{Z}_2)$, there exists a homology class $\xi_q \in \overset{\circ}{H}_q(B; \boldsymbol{Z}_2)$ such that $\langle \xi_q, s_2^q(B)\rangle \neq 0$. By virtue of Proposition (4.4), $\bar{\Omega}(\phi, \psi)\xi_q = s^p(A) \neq 0$; consequently, $\phi(A) \cap \psi(B) \neq \varnothing$, by Proposition (4.3).

Theorem (4.5) implies Theorem (3.2) by letting $A = \tilde{X}^*$, $B = \tilde{Y}^*$ and by defining $\phi: A \to S^n$, $\psi: B \to S^n$ by the formulas

$$\phi(x_1, x_2) = \frac{f(x_1) - f(x_2)}{|f(x_1) - f(x_2)|}, \qquad \psi(y_1, y_2) = \frac{g(y_1) - g(y_2)}{|g(y_1) - g(y_2)|}$$

for $(x_1, x_2) \in \tilde{X}^*$, $(y_1, y_2) \in \tilde{Y}^*$.

In particular, Theorem (4.5) implies that if A, B are two antipodal subsets of S^n such that $\mathrm{Ind}(A) + \mathrm{Ind}_2(B) \geqq n$, then $A \cap B \neq \varnothing$; here $\phi: A \to S^n$ and $\psi: B \to S^n$ are the inclusion maps. This corollary is stronger than [2, Theorem 2].

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