COINCIDING DIRECTIONS FOR IMBEDDINGS
OF SPACES

JAN W. JAWOROWSKI

W. T. Wu [4] and [5] and A. Shapiro [3] defined obstruction classes \( w^i(X) \) to the existence of an imbedding \( f: X \to \mathbb{R}^n \) of a topological space \( X \) in the euclidean \( n \)-space \( \mathbb{R}^n \). The Wu-Shapiro classes \( w^i \) are particular cases of the Smith classes defined for a space with an involution. We recall briefly the definition.

1. Equivariant cohomology. Let \( A \) be a space with a continuous fixed point free involution \( \alpha: A \to A \). Consider the two possible actions of \( \alpha \) on the coefficient group \( \mathbb{Z} \) of integers: the trivial action and the nontrivial one. Let \( \hat{H}_+^i(A) \) (resp. \( \hat{H}_-^i(A) \)) denote the equivariant (resp. residual) \( i \)th cohomology group of \( A \) for the trivial action; and let \( \hat{H}_-^i(A) \) (resp. \( \hat{H}_-^i(A) \)) denote the corresponding equivariant (resp. residual) cohomology groups for the nontrivial action of \( \alpha \) on \( \mathbb{Z} \) (the singular homology is considered throughout; see [1]). The groups \( \hat{H}_+^i(A) \) and \( \hat{H}_-^i(A) \) can also be described as the corresponding cohomology groups of the orbit space \( A/\alpha \), with the constant and twisted coefficients \( \{\mathbb{Z}\} \), respectively.

There are exact sequences:

\[
\cdots \to \hat{H}^{i-1}(A) \to \hat{H}_+^{i-1}(A) \to \hat{H}_+^i(A) \to \hat{H}^i(A) \to \cdots, \\
\cdots \to \hat{H}^{i-1}(A) \to \hat{H}_-^{i-1}(A) \to \hat{H}_-^i(A) \to \hat{H}^i(A) \to \cdots.
\]

Consider the singular chain complex \( C(A) \) of \( A \) and the chain maps 
\( 1 + (-1)^i \alpha \), where \( \alpha_i: C_i(A) \to C_i(A) \) is induced by \( \alpha \). They induce homomorphisms

\[
t^i: \hat{H}_-^i(A) \to \hat{H}_+^i(A) \quad \text{if } i \text{ is even},
\]

\[
t^i: \hat{H}_+^i(A) \to \hat{H}_-^i(A) \quad \text{if } i \text{ is odd}.
\]

The homomorphisms \( t^i \) are known to be isomorphisms (see [5]). Therefore we shall identify the groups \( \hat{H}_+^i(A) \) and \( \hat{H}_+^i(A) \), if \( i \) is even, and the groups \( \hat{H}_-^i(A) \) and \( \hat{H}_-^i(A) \), if \( i \) is odd, under the isomorphisms

Received by the editors January 17, 1967.

1 Research supported by NSF Grant GP-6015.
\( t^i \) and denote them by \( \hat{H}^i(A) \). Thus \( \hat{H}^i \) denotes \( \hat{H}^i_+ = \hat{H}^i_- \), if \( i \) is even, and \( \hat{H}^i_+ = \hat{H}^i_- \), if \( i \) is odd.

Let \( \eta^p: \hat{H}^0(A) \to \hat{H}^p(A) \) denote the homomorphism defined by
\[
\eta^p = \delta^{p-1} \circ (t^{p-1})^{-1} \circ \cdots \circ \delta^0 \circ (t^0)^{-1}.
\]

In particular, \( \hat{H}^0(A) = \hat{H}^0_+(A) \) and it contains the constant unit cohomology class \( 1 \in \hat{H}^0(A) \). The cohomology class
\[
s^p(A) = \eta^p(1) \in \hat{H}^p(A)
\]
is called the \( p \)th Smith class of \( A \). The index \( \text{Ind}(A) \) of \( A \) is defined to be the maximal integer \( p \) such that \( s^p(A) \neq 0 \) (or \( \infty \) if \( s^p(A) \neq 0 \) for all integers \( p \geq 0 \)).

If \( A = S^n \) and \( \alpha: S^n \to S^n \) is the antipodal involution, then \( H^i(S^n) \cong \mathbb{Z}_2 \), for \( 0 < i \leq n \), generated by \( s^i(S^n) \); and \( \text{Ind}(S^n) = n \).

We use a similar notation for homology. Then we have dual isomorphisms
\[
t_i: \hat{H}_i(A) \to \hat{H}_i(A) \quad \text{if } i \text{ is even},
\]
\[
t_i: \hat{H}_i(A) \to \hat{H}_i(A) \quad \text{if } i \text{ is odd};
\]
they are induced by the chain map \( 1 + (-1)^{i-1}\alpha_i: C_i(A) \to C_i(A) \). We again identify the corresponding groups under \( t_i \) and denote them by \( \hat{H}_i(A) \). Thus \( \hat{H}_i \) denotes \( \hat{H}^-_i = \hat{H}^+_i \), if \( i \) is even, and \( \hat{H}^+_i = \hat{H}^-_i \), if \( i \) is odd.

It is easy to verify that the Kronecker index pairing \( \langle , \rangle: \hat{H}_i(A) \otimes \hat{H}_j(A) \to \mathbb{Z}_2 \) is defined.

If \( A \) and \( B \) are two spaces with fixed point free involutions and \( \phi: A \to B \) is an involution preserving map, then \( \phi^* s^p(B) = s^p(A) \) and thus \( \text{Ind}(A) \leq \text{Ind}(B) \).

A space \( A \) is said to be \((-1)\)-acyclic if it is nonempty; it is said to be \( p \)-acyclic, \( p \geq 0 \), if its cohomology groups \( H^i(A) \) are isomorphic to those of a one-point space in dimensions \( i \leq p \).

Observe that if \( A \) is \( p \)-acyclic, then the connecting homomorphisms \( \delta^{i-1} \) in sequences (1.1) and (1.2) are isomorphisms, for \( i \leq p \), and monomorphisms for \( i = p + 1 \). Thus if \( A \) is \( p \)-acyclic, then \( \text{Ind}(A) \geq p + 1 \).

2. **Case mod 2.** We obtain a simplified version of the above theory if we use the group \( \mathbb{Z}_2 \) as coefficient group. In this case there is only one action of \( \alpha \) on the coefficient group, the trivial one, and sequences (1.1) and (1.2) coincide. We denote the resulting homomorphisms analogous to \( \eta^p \) by
\[ \eta_2: \tilde{H}^0(A; Z_2) \to \tilde{H}^p(A; Z_2); \]

the Smith class mod 2 by \( s^2_2(A) = \eta_2^p(1) \); and the index mod 2 by \( \text{Ind}_2(A) \).

Observe that we have the following commutative diagram (for a nonempty space \( A \)):

\[
\begin{array}{ccc}
\tilde{H}^0(A) & \xrightarrow{\eta^p} & \tilde{H}^p(A) \\
\downarrow & & \downarrow \\
\tilde{H}^0(A; Z_2) & \xrightarrow{\eta_2} & \tilde{H}^p(A; Z_2).
\end{array}
\]

Here the vertical arrows denote the reduction of the coefficients mod 2. Since the homomorphism \( \tilde{H}^0(A) \to \tilde{H}^0(A; Z_2) \) carries the unit cohomology class of \( \tilde{H}^0(A) \) to the unit cohomology class of \( \tilde{H}^0(A; Z_2) \), it follows that

(2.1) \[ \text{Ind}_2(A) \leq \text{Ind}(A). \]

Remark (2.2). It is easily seen that \( \text{Ind}_2(A) \) coincides with the index of \( A \) as defined in [6]. Also, \( \text{Ind}_2(A) \geq p \) if and only if \( A \) contains a \((p, \alpha)\)-system in the sense of [2].

3. Imbeddings into \( R^n \) and Wu-Shapiro classes. Let \( X \) be a space and let \( \tilde{X}^* \) denote the deleted square \( \tilde{X} \times \tilde{X} - \Delta \), where \( \Delta \) is the diagonal. Let \( \alpha: \tilde{X}^* \to \tilde{X}^* \) be the involution permuting the coordinates \( \alpha(x_1, x_2) = (x_2, x_1) \). Then \( \omega^p(X) = s^p(\tilde{X}^*) \in \tilde{H}^p(\tilde{X}^*) \) are the Wu-Shapiro classes of \( X \). We also denote

\[ I(X) = \text{Ind}(\tilde{X}^*), \quad I_2(X) = \text{Ind}_2(\tilde{X}^*). \]

If \( f: X \to R^n \) is a topological imbedding, then the map \( \phi: \tilde{X}^* \to S^{n-1} \) defined by

\[ \phi(x_1, x_2) = (f(x_1) - f(x_2))/ \sqrt{f(x_1)^2 - f(x_2)^2} \]

is involution preserving (for the antipodal involution on \( S^{n-1} \)). Since \( S^{n-1} \) is a deformation retract of \( \tilde{S}^{n-1} \) by a deformation which preserves the involution \( \alpha \), it follows that \( I(X) < n \). Thus we have

**Theorem (3.1) (Wu-Shapiro).** *If \( X \) is imbeddable in \( R^n \), then \( I(X) < n \), i.e. \( \omega^n(X) = 0 \).*

This means that \( \omega^n(X) \) is an obstruction to the existence of an imbedding \( f: X \to R^n \). Wu [5] and Shapiro [3] proved that if \( X \) is a finite polyhedron and \( n > 2 \), then the condition \( \omega^{2n}(X) = 0 \) is also sufficient for the imbeddability of \( X \) in \( R^{2n} \).
Suppose now that \( f: X \to R^n \) and \( g: Y \to R^n \) are (topological) imbeddings of spaces \( X \) and \( Y \) in \( R^n \). We say that \( f \) and \( g \) have a coinciding direction if there exist two pairs of distinct points \( x_1, x_2 \in X \), \( x_1 \neq x_2 \); \( y_1, y_2 \in Y \), \( y_1 \neq y_2 \), such that the lines 
\[
\overline{f(x_1)f(x_2)} \quad \text{and} \quad \overline{g(x_1)g(x_2)}
\]
are parallel.

The purpose of this paper is to prove the following

**Theorem (3.2).** If \( I(X) + I_2(Y) \geq n \), then any imbeddings \( f: X \to R^{n+1} \), \( g: Y \to R^{n+1} \) have a coinciding direction.

Theorem (3.1) can be obtained as a special case of (3.2): if there exists an imbedding \( f: X \to R^n \subset R^{n+1} \), let \( Y \) be a space consisting of two points, and let \( g: Y \to R^{n+1} \) be an imbedding of \( Y \) into a line in \( R^{n+1} \) orthogonal to \( R^n \). Then \( f \) and \( g \) have no coinciding direction and thus \( I(X) + I_2(Y) = I(X) < n \).

Actually, a somewhat stronger result than (3.2) will be proved; but the question whether the condition \( I(X) + I(Y) \geq n \) is also sufficient for the existence of a coinciding direction for \( f \) and \( g \) remains open.

**Example (3.3).** For \( X = Y = S^1 \), we have \( I(S^1) = I_2(S^1) = 1 \). Thus any two simple closed curves in \( R^3 \) have a pair of parallel chords. Similarly, if \( Y \) is a triod (a space homeomorphic to the letter \( Y \)), then \( I(Y) = I_2(Y) = 1 \). Thus any two triods or a triod and a simple closed curve in \( R^3 \) have coinciding directions.

4. **Involution preserving maps into \( S^n \) and the intersection homomorphism.** Let \( A \) and \( B \) be spaces with involutions \( \alpha: A \to A \), \( \beta: B \to B \), and let \( \phi: A \to S^n \), \( \psi: B \to S^n \) be involution preserving maps (for the antipodal involution of \( S^n \)). Let \( p \) and \( q \) be integers such that \( p + q = n \). Then \( \phi, \psi \) define a natural intersection homomorphism
\[
\Omega(\phi, \psi): \hat{H}_q(B) \to \hat{H}_p(A).
\]

The definition of \( \Omega(\phi, \psi) \) can be sketched as follows. Let \( T \) be an antipodal cell decomposition of \( S^n \). For each homology class \( \xi_q \in \hat{H}_q(B) \), the class \( \psi_*(\xi_q) \) may be represented by an (equivariant) cycle \( z_q \) of \( T \). The cycle \( z_q \) defines a homomorphism \( u^p: C_p(A) \to \mathbb{Z} \) as follows: for each chain \( c_p \in C_p(A) \) we can place \( z_q \) and \( c_p \) in general position and define \( u^p(c_p) \) to be the intersection number of \( z_q \) and \( c_p \). Since \( z_q \) is equivariant, the cochain \( u^p \) is also equivariant and it is actually a cocycle. We define \( \Omega(\phi, \psi)\xi_q \) to be the cohomology class represented by \( u^p \).
By considering two dual antipodal cell decompositions $T$ and $T^*$ of $S^n$, it can be seen that the (unique) isomorphism $\tilde{H}_q(S^n) \cong \tilde{H}^p(S^n) \cong \mathbb{Z}_2$ is just the homomorphism

$$\omega = \Omega(1_{S^n}, 1_{S^n}) : \tilde{H}_q(S^n) \to \tilde{H}^p(S^n)$$

which we denote briefly by $\omega$; $1_{S^n}$ is the identity map of $S^n$; thus $\omega$ is the Poincaré isomorphism. It follows that the intersection homomorphism $\Omega(\phi, \psi)$ can be defined equivalently by the following commutative diagram

$$
\begin{array}{ccc}
\tilde{H}_q(B) & \xrightarrow{\Omega(\phi, \psi)} & \tilde{H}^p(A) \\
\downarrow{\psi_*} & & \downarrow{\phi_*} \\
\tilde{H}_q(S^n) & \xrightarrow{\omega} & \tilde{H}^p(S^n),
\end{array}
$$

i.e. $\Omega(\phi, \psi) = \phi^* \omega \psi_*$.

**Proposition (4.1).** If $A$ and $B$ are spaces with involutions and $\phi : A \to S^n$, $\psi : B \to S^n$ are involution preserving maps such that $\phi(A) \cap \psi(B) = \emptyset$, then $\Omega(\phi, \psi) = 0$.

For $\Omega(\phi, \psi)$ is defined by means of intersection numbers.

**Proposition (4.2).** If $\phi, \psi$ are as in Proposition (4.1) and $\xi_q \in \tilde{H}_q(B)$ is a homology class such that the Kronecker index $\langle \xi_q, s^q(B) \rangle \neq 0$, then $\Omega(\phi, \psi) \xi_q = s^p(A)$.

**Proof.** By the naturality of the Kronecker index and the Smith classes, it follows that $\langle \psi_* (\xi_q), s^q(S^n) \rangle \neq 0$; but this means that $\psi_*(\xi_q)$ is the unique nonzero element of $\tilde{H}_q(S^n)$. Consequently, $\omega \psi_*(\xi_q)$ is the unique nonzero element of $\tilde{H}^p(S^n)$, i.e. the Smith class $\omega \psi_*(\xi_q) = s^p(S^n)$. It follows that $\Omega(\phi, \psi) = \phi^* \omega \psi_*(\xi_q) = s^p(A)$.

Due to the fact that there is a unique isomorphism $\iota : \tilde{H}_q(S^n; \mathbb{Z}_2) \cong H_q(S^n) \cong \mathbb{Z}_2$, we can define an intersection homomorphism

$$\tilde{\Omega}(\phi, \psi) = \phi^* \omega \psi_* : \tilde{H}_q(B; \mathbb{Z}_2) \to \tilde{H}^p(A).$$

Propositions corresponding to Propositions (4.1) and (4.2) hold true:

**Proposition (4.3).** Under the assumptions of Proposition (4.1), $\Omega(\phi, \psi) = 0$.  

**Proposition (4.4).** If \( \xi_q \in H_q(B; \mathbb{Z}_2) \) is a homology class such that the Kronecker index \( \langle \xi_q, s^p(B) \rangle \neq 0 \), then \( \Omega(\phi, \psi) \xi_q = s^p(A) \).

For we have again \( \langle \psi_*(\xi_q), s^Q(S^n) \rangle \neq 0 \); it follows that \( \psi_*(\xi_q) \) is the nonzero element of \( H_q(S^n; \mathbb{Z}_2) \), hence \( \psi_\ast \) is the nonzero element of \( H_q(S^n) \) and \( \omega \psi_\ast(\xi_q) = s^p(S^n) \). Thus \( \Omega(\phi, \psi) \xi_q = s^p(A) \).

**Theorem (4.5).** If \( A \) and \( B \) are spaces with involutions such that \( \text{Ind}(A) + \text{Ind}_2(B) \geq n \) and \( \phi: A \to S^n, \psi: B \to S^n \) are involution preserving maps, then \( \phi(A) \cap \psi(B) \neq \emptyset \).

**Proof.** There exist integers \( p, q \) with \( p + q = n \) such that \( s^p(A) \neq 0, s^2(B) \neq 0 \). Since \( H_q(B; \mathbb{Z}_2) = \text{Hom}(H_q(B; \mathbb{Z}_2), \mathbb{Z}_2) \), there exists a homology class \( \xi_q \in H_q(B; \mathbb{Z}_2) \) such that \( \langle \xi_q, s^Q(B) \rangle \neq 0 \). By virtue of Proposition (4.4), \( \Omega(\phi, \psi) \xi_q = s^p(A) \neq 0 \); consequently, \( \phi(A) \cap \psi(B) \neq \emptyset \), by Proposition (4.3).

Theorem (4.5) implies Theorem (3.2) by letting \( A = \tilde{X}^*, B = \tilde{Y}^* \) and by defining \( \phi: A \to S^n, \psi: B \to S^n \) by the formulas

\[
\phi(x_1, x_2) = \frac{f(x_1) - f(x_2)}{|f(x_1) - f(x_2)|}, \quad \psi(y_1, y_2) = \frac{g(y_1) - g(y_2)}{|g(y_1) - g(y_2)|}
\]

for \((x_1, x_2) \in \tilde{X}^*, (y_1, y_2) \in \tilde{Y}^*\).

In particular, Theorem (4.5) implies that if \( A, B \) are two antipodal subsets of \( S^n \) such that \( \text{Ind}(A) + \text{Ind}_2(B) \geq n \), then \( A \cap B \neq \emptyset \); here \( \phi: A \to S^n \) and \( \psi: B \to S^n \) are the inclusion maps. This corollary is stronger than [2, Theorem 2].

**References**


**Indiana University**