# COHOMOLOGY OF $\Omega(G/U)$

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Let $G$ be a compact connected Lie group and $U \subset G$ a closed connected subgroup. Our objective is to determine the cohomology of the space of loops on the homogenous space $G/U$. Our main tool is a spectral sequence of Eilenberg and Moore ([3], [4], [7], [8]) from which we obtain the following:

THEOREM P1. *Let $p$ be a prime and $T \subset G$ a maximal torus of $G$. Assume that $H^*(G, Z_p)$ is an exterior algebra on odd dimensional generators. Then there exists a filtration $\{F^{-n}H^*(\Omega(G/T), Z_p); n \geqq 0\}$ such that*

$$E^0H^*(\Omega(G/T), Z_p) = \mathrm{Tor}_{H^*(G/T, Z_p)}(Z_p, Z_p)$$

*as Hopf algebras.*

THEOREM Q1. *Let $Q$ be the rational numbers and let $G$, $T$ be as in Theorem* P1, *then there is an isomorphism of algebras*

$$H^*(\Omega(G/T), Q) = \mathrm{Tor}_{H^*(G/T, Q)}(Q, Q).$$

THEOREM P2. *Let $G$ be simply connected and $U$ a closed connected subgroup and $i$: $U \subset G$ the inclusion map. Consider $H^*(U, Z_p)$ as an $H^*(G, Z_p)$ module via the map $i^*: H^*(G, Z_p) \to H^*(U, Z_p)$. Then if $H^*(G, Z_p)$ is an exterior algebra on odd dimensional generators there is a filtration $\{F^{-n}H^*(\Omega(G/U), Z_p); n \geqq 0\}$ such that*

$$E^0H^*(\Omega(G/U), Z_p) = \mathrm{Tor}_{H^*(G, Z_p)}(Z_p, H^*(U, Z_p))$$

*as Hopf algebras.*

THEOREM Q2. *Let $Q$ be the rational numbers, $G$, $U$ as in Theorem* P2. *Then there is an isomorphism of algebras*

$$H^*(\Omega(G/U), Q) = \mathrm{Tor}_{H^*(G, Q)}(Q, H^*(U, Q)).$$

1. **Algebra.** Throughout this section $k$ will denote a fixed field. All modules will be graded $k$-modules of finite type, all algebras will be connected associative and commutative; $\otimes$ means $\otimes_k$. Our notation will be that of [5].

DEFINITION. If $A$ is an algebra, $a_1, \cdots, a_n \in A$, then we say that $a_1, \cdots, a_n$ is an ESP-sequence if $a_i$ is not a zero divisor in the algebra

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$A/(a_1, \cdots, a_{i-1})$ for $i=1, \cdots, n-1$. (Here $(a_1, \cdots, a_{i-1})$ denotes the ideal generated by $a_1, \cdots, a_{n-1}$.)

DEFINITION. An algebra $\Lambda$ is called a saturated algebra if it is the quotient of a polynomial algebra $P[x_1, \cdots, x_n]$ by an ESP-sequence $r_1, \cdots, r_n$.

It is easy to see that if $\Lambda$ is a saturated algebra then it can be presented as a quotient of a polynomial algebra $P[y_1, \cdots, y_m]$ by an ESP-sequence $s_1, \cdots, s_m$ where no $s_i$ is indecomposable. We will always assume that our presentations are minimal in this sense.

Our objective in this section is to compute $\operatorname{Tor}_\Lambda(k, k)$ when $\Lambda$ is a saturated algebra. To this end let

$$\Lambda = P[x_1, \cdots, x_n]/(r_1, \cdots, r_n)$$

be a minimal presentation of $\Lambda$. Let

$$\mathcal{E} = \Lambda \otimes E[u_1, \cdots, u_n]$$

where $\deg u_i = (-1, \deg x_i)$. We give $\mathcal{E}$ the structure of a complex by defining a differential $\partial$ by the formulas

$$\partial(\lambda \otimes 1) = 0, \quad \partial(\lambda \otimes u_i) = \lambda x_i \otimes 1, \quad i = 1, \cdots, n, \quad \lambda \in \Lambda,$$

and requiring $\partial$ to be a derivation of algebras. An easy calculation [**10**] shows

$$H^*(\mathcal{E}) = E[v_1, \cdots, v_n], \qquad \deg v_i = (-1, \deg r_i).$$

Now form the algebra

$$\mathcal{R} = \mathcal{E} \otimes \Gamma[z_1, \cdots, z_n], \qquad \deg z_i = (-2, \deg r_i)$$

and define a derivation $d$ on $\mathcal{R}$ as follows: for each $i$ choose a representative cocycle $\bar{v}_i \in \mathcal{E}$ representing $v_i \in H(\mathcal{E})$. Now define $d$ by

$$d \mid \mathcal{E} = \partial, \qquad d(z_i) = \bar{v}_i, \quad i = 1, \cdots, n,$$

and requiring that $d$ be a derivation of algebras with divided powers. (Note that if the characteristic of $k$ is zero then $\Gamma[z_1, \cdots, z_n] = P[z_1, \cdots z_n]$.) As in [**10**] we can now show that $\mathcal{R} \to k$ is a $\Lambda$-free resolution of $k$ and hence

PROPOSITION 1.1. $\operatorname{Tor}_\Lambda(k, k) = E[u_1, \cdots, u_n] \otimes \Gamma[z_1, \cdots, z_n]$ *as a Hopf algebra over* $k$. $\square$

2. **Torus subgroups.** Let $G$ be a compact connected Lie group and $T$ a maximal torus of $G$. The inclusion $T \subset G$ induces a fibre bundle

$$\rho\colon G/T \to B_T \xrightarrow{\rho} B_G$$

where $B_T$ and $B_G$ are classifying spaces for $T$ and $G$ respectively. Let $k=Z_p$ or $Q$ and assume that $H^*(G, k)$ is an exterior algebra on odd dimensional generators (this is no restriction if $k=Q$). We then have the fundamental result of Borel [2]:

THEOREM ([2], [1]). *$H^*(B_G, k)=P[y_1, \cdots, y_n]$ $n=\operatorname{rank} G$, $\rho^*(y_1), \cdots, \rho^*(y_n)$ is an ESP-sequence in $H^*(B_T, k)$ and $H^*(G/T, k) = H^*(B_T, k)/\!\!/\rho^*$. Since $H^*(B_T, k)=P[x_1, \cdots, x_n]$ $\deg x_i=2$ it follows that $H^*(G/T, k)$ is a saturated algebra.*

PROOF OF THEOREM P1. Let $\{E_r, d_r\}$ denote the Eilenberg-Moore spectral sequence [3], [4], [8] of the pathspace fibration

$$\Omega(G/T) \to P(G/T) \to G/T$$

we then have

$$E_r \Rightarrow H^*(\Omega(G/T), Z_p), \qquad E_2 = \operatorname{Tor}_{H^*(G/T, Z_p)}(Z_p, Z_p).$$

We will show that $E_2=E_\infty$ which will complete the proof. From §1 we see that

$$E_2 = E[u_1, \cdots, u_m] \otimes \Gamma[z_1, \cdots, z_m]$$

where $m \leqq n$, $\deg u_i=(-1, 2)$, $\deg z_i=(-2, 2t_i)$, $t_i \geqq 2$. Now $\{E_r, d_r\}$ is a spectral sequence of Hopf algebras. Let $s$ be the smallest integer $\geqq 2$ such that $d_s \neq 0$. Then $E_2=E_s$. Let $a \in E_2$ of minimal degree such that $d_s(a) \neq 0$. Then we may assume that $a$ is indecomposable, and therefore $\deg a=(-2p^q, 2tp^q)$, $t>1$. A simple computation shows that $d_s(a)$ must be primitive. Since the total degree of $a$ is even it follows that $d_s(a)$ is a primitive of odd total degree and hence $\deg d_s(a) = (-1, 2)$. Since $\deg d_s=(s, 1-s)$ we see that $s=2p^q-1$ and hence we have $\deg d_s(a)=(-1, 2+2(t-1)p^q)$. But $t>1$ and hence $2+2(t-1)p^q>2$ which contradicts our previous calculation of $\deg d_s(a)$. Therefore our assumption that $d_s(a) \neq 0$ must be false and hence $E_2=E_\infty$. $\square$

PROOF OF THEOREM Q1. We recall [2] that $H^*(G, Q)$ is an exterior algebra on odd dimensional generators. By the argument above we see the Eilenberg-Moore spectral sequence of the fibration

$$\Omega G/T \xrightarrow{j} P(G/T) \xrightarrow{i} G/T$$

with $Q$ coefficients has $E_2=E_\infty$.

But $E_2=E[u_1, \cdots, u_m] \otimes P[z_1, \cdots, z_m]$ is a free commutative algebra and hence the extension as algebras is trivial. $\square$

3. **A fibration.** Let $U$ be a closed connected subgroup of $G$ and $i: U \subset G$ the inclusion map. By a standard construction [9, p. 99] we can replace $i$ by an equivalent fibre mapping whose fibre is $\Omega(G/U)$. This gives a multiplicative fibre map

$$\Omega(G/U) \to U \to G.$$

If $G$ is simply connected then this fibring has for each field $k$ an Eilenberg-Moore spectral sequence $\{\tilde{E}_r, \tilde{d}_r\}$ with

$$\tilde{E}_r \Rightarrow H^*(\Omega(G/U), k), \qquad \tilde{E}_2 = \mathrm{Tor}_{H^*(G,k)}(k, H^*(U, k)).$$

Henceforth we shall assume that $k = Z_p$ or $Q$ and that $H^*(G, k)$ is an exterior algebra on odd dimensional generators. From [2] it follows that $H^*(G, k)$ is primitively generated and hence cocommutative. Let $H^*(G, k) \diagdown i^*$ denote the kernal in the category of bicommutative Hopf algebras of the map $i^*: H^*(G, k) \to \mathrm{im}\, i^*$ (see [6], [7], [8]). Then as in [7], [8] we can show

PROPOSITION.

$$\mathrm{Tor}_{H^*(G,k)}(k, H^*(U, k)) = H^*(U, k) /\!\!/ i^* \otimes \mathrm{Tor}_{H^*(G,k) \diagdown i^*}(k,k)$$

*as Hopf algebras.* □

4. **The simply connected case.** Before proving Theorem P2 we state a preliminary technical result whose proof we postpone until later.

PROPOSITION.

$$\ker\{j^*: H^*(U, Z_p) \to H^*(\Omega(G/U), Z_p)\} = \overline{\mathrm{im}\, i^*} \cdot H^*(U, Z_p).$$

PROOF OF THEOREM P2. By the Proposition of §3 we have

$$E_2 = H^*(U, Z_p) /\!\!/ i^* \otimes \mathrm{Tor}_{H^*(G,Z_p) \diagdown i^*}(Z_p, Z_p).$$

From Borel's structure theorem for Hopf algebras over $Z_p$ [6, 7.11] we deduce that $H^*(G, Z_p) \diagdown i^* = E[u_1, \cdots, u_m]$, degree $u_i$ odd, and hence we deduce that as Hopf algebras

$$E_2 = H^*(U, Z_p) /\!\!/ i^* \otimes \Gamma[y_1, \cdots, y_m]$$

where deg $y_i = (-1, \deg u_i)$. Suppose that $E_2 \neq E_\infty$. A repetition of the argument in Theorem P1 shows that there is an $x \in E_2$ such that $d_r(x) \neq 0 \in E_2^{0,*}$. The map $j^*$ is given by the composition

$$H^*(U, Z_p) \to H^*(U, Z_p) /\!\!/ i^* = E_2^{0,*} \xrightarrow{\epsilon} E_\infty^{0,*} \subset H^*(\Omega(G/U), Z_p)$$

and hence from the Proposition above it follows that the edge map $\epsilon$ is a monomorphism. But this contradicts the fact that $d_r(x)\neq 0$. Therefore it follows that $E_2=E_\infty$. $\square$

PROOF OF PROPOSITION. Consider the fibration

$$\Omega(G)\to\Omega(G/U)\xrightarrow{j}U$$

obtained by replacing $j$ by a fibre map. Let $\{\hat{E}^r, \hat{d}^r\}$ denote the homology Serre spectral sequence of this fibration. Consider the diagram

$$\begin{array}{ccc}
\Omega(G) & \to & \Omega(G) \\
\downarrow & & \downarrow \\
\Omega(G/U) & \to & P(G) \\
\downarrow & & \downarrow \\
U & \to & G
\end{array}$$

and recall that we have by [6, 4.4]

(1) $H_*(G, Z_p)=E[x_1, \cdots, x_n]$

(2) $H_*(U, Z_p)=E[y_1, \cdots, y_t]\otimes M$ where

$$\begin{aligned} i_*(y_i) &= x_i, \qquad 1\leqq i\leqq t, \\ i_*/M &= 0, \end{aligned}$$

and

(3) $H_*(\Omega(G), Z_p)=P[z_1, \cdots, z_n]$ with $\tau x_i=z_i$ where $\tau$ is the transgression in the path space fibring over $G$. A standard spectral sequence argument now shows that $\hat{E}^\infty=M\otimes P[z_{t+1}, \cdots, z_n]$ and since the map $j_*$ is the composition $H_*(U, Z_p)\to\hat{E}^2_{0,*}\to\hat{E}^\infty_{0,*}\subset H_*(\Omega(G/U), Z_p)$ the result follows by duality. $\square$

The proof of Theorem Q2 is an easy exercise and is left to the reader.

In closing we note that if $G$ is simply connected and $T$ is any torus in $G$ then the inclusion map $i$: $T\subset G$ is null homotopic and hence standard homotopy theory shows that $\Omega(G/T)$ has the homotopy type of $T\times\Omega(G)$, and hence the study of torus subgroups in the simply connected case is trivial.

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