CHROMATIC NUMBER OF CARTESIAN SUM OF TWO GRAPHS

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In this note, we will consider the class of all finite undirected graphs with simple edges and no loops [1]. Let $G, G_1, G_2$ denote graphs. Let $\omega (G) =$ the number of vertices in $G$, $\beta (G) =$ the independence number of $G$, $\chi (G) =$ the chromatic number of $G$, $G_1 \oplus G_2 =$ the Cartesian sum of $G_1$ and $G_2$ [1]. We shall prove the following

**Theorem.** $\omega (G_1)\omega (G_2) / \beta (G_1)\beta (G_2) \leq \chi (G_1 \oplus G_2) \leq \chi (G_1)\chi (G_2)$.

Moreover, an example is given to show that the inequality $\chi (G_1 \oplus G_2) < \chi (G_1)\chi (G_2)$ in fact occurs.

The proof is based on the following two lemmas.

**Lemma 1.** $\beta (G_1 \oplus G_2) = \beta (G_1)\beta (G_2)$.

**Proof.** If $G$ is a graph, we denote by $V(G)$ the set of vertices of $G$ and by $E(G)$ the set of edges of $G$. We say that a subset $S \subseteq V(G)$ is independent if for any $a, b \in S$, $(a, b) \notin E(G)$. If $S$ is a set, we denote by $| S |$ the number of elements in $S$. Now, let $S_i \subseteq V(G_i)$ $(i = 1, 2)$ be independent sets such that $| S_i | = \beta (G_i)$. If $S = \{ a_1 \oplus a_2 : a_1 \in S_1$ and $a_2 \in S_2 \}$, then $S$ is an independent set of $G_1 \oplus G_2$. Therefore, $\beta (G_1 \oplus G_2) \geq | S | = \beta (G_1)\beta (G_2)$.

Suppose $T \subseteq V(G_1 \oplus G_2)$ is an independent set such that $| T | = \beta (G_1 \oplus G_2)$. For $a \in V(G_1)$, let $T(a) = \{ b \in V(G_2) : a \oplus b \in T \}$. Then for each $a \in V(G_1)$, $T(a)$ is an independent set of $G_2$. Therefore, for each $a \in V(G_1)$, $| T(a) | \leq \beta (G_2)$. Clearly, $| T | = \sum_{a \in V(G_1)} | T(a) |$. But $| T(a) | = 0$ except for those $a$ in an independent set of $G_1$. Hence $\beta (G_1 \oplus G_2) = | T | \leq \beta (G_1)\beta (G_2)$. This shows that $\beta (G_1 \oplus G_2) = \beta (G_1)\beta (G_2)$.

**Lemma 2.** $\chi (G_1)\chi (G_2) \leq \chi (G_1 \oplus G_2)$.

**Proof.** Let $G, H$ be graphs. A homomorphism $f : G \rightarrow H$ is a map $f : V(G) \rightarrow V(H)$ such that if $(a, b) \in E(G)$, then $(f(a), f(b)) \in E(H)$. We say that a homomorphism $f : G \rightarrow H$ is surjective if $f : V(G) \rightarrow V(H)$ is surjective. If we denote by $K_n$ the complete graph of $n$ vertices, then the chromatic number of a graph $G$ can be defined as the smallest integer $n$ such that there exists a surjective homomorphism $f : G \rightarrow K_n$. Now let $m = \chi (G_1)$ and $n = \chi (G_2)$ and let $f_1 : G_1 \rightarrow K_m$ and $f_2 : G_2 \rightarrow K_n$ be surjective homomorphisms. If we define

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by

\[ f_1 \oplus f_2: G_1 \oplus G_2 \rightarrow K_m \oplus K_n \]

then \( f_1 \oplus f_2 \) is obviously a surjective homomorphism. Since \( K_m \oplus K_n \cong K_{mn} \), we obtain \( K(G_1 \oplus G_2) \leq K(G_1) \cdot K(G_2) \).

**Proof of the Theorem.** We only need to show that

\[ (\alpha(G_1) \alpha(G_2) / \beta(G_1) \beta(G_2)) \leq K(G_1 \oplus G_2). \]

By [1, Theorem 14.1.1, p. 225], we have

\[ (\alpha(G_1 \oplus G_2) / \beta(G_1 \oplus G_2)) \leq K(G_1 \oplus G_2). \]

By Lemma 1, \( \beta(G_1 \oplus G_2) = \beta(G_1) \beta(G_2) \). It is obvious that \( \alpha(G_1 \oplus G_2) = \alpha(G_1) \alpha(G_2) \). Therefore, \( (\alpha(G_1) \alpha(G_2) / \beta(G_1) \beta(G_2)) \leq K(G_1 \oplus G_2) \).

Finally, we present an example to show that the inequality \( K(G_1 \oplus G_2) < K(G_1) \cdot K(G_2) \) actually occurs.

Let \( G_1 \) be the graph with

\[ V(G_1) = \{1, 2, 3, 4, 5\}, \quad E(G_1) = \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 1)\}. \]

Let \( G_2 \) be the same graph with different labeling of the vertices, say,

\[ V(G_2) = \{1', 2', 3', 4', 5'\}, \quad E(G_2) = \{(1', 2'), (2', 3'), (3', 4'), (4', 5'), (5', 1')\}. \]

Clearly, \( K(G_1) = K(G_2) = 3 \). We now define a surjective homomorphism \( f: G_1 \oplus G_2 \rightarrow K_8 \), where \( V(K_8) = \{k_1, k_2, \ldots, k_8\} \), by

- \( f^{-1}[k_1] = \{1 \oplus 1', 1 \oplus 3', 3 \oplus 1', 3 \oplus 3'\} \),
- \( f^{-1}[k_2] = \{1 \oplus 2', 1 \oplus 4', 3 \oplus 2', 3 \oplus 4'\} \),
- \( f^{-1}[k_3] = \{2 \oplus 1', 2 \oplus 3', 4 \oplus 1', 4 \oplus 3'\} \),
- \( f^{-1}[k_4] = \{2 \oplus 2', 2 \oplus 4', 4 \oplus 2', 4 \oplus 4'\} \),
- \( f^{-1}[k_5] = \{3 \oplus 5', 5 \oplus 3', 5 \oplus 5'\} \),
- \( f^{-1}[k_6] = \{2 \oplus 5', 5 \oplus 2'\} \),
- \( f^{-1}[k_7] = \{1 \oplus 5', 4 \oplus 5'\} \),
- \( f^{-1}[k_8] = \{5 \oplus 1', 5 \oplus 4'\} \).

The example shows that \( K(G_1 \oplus G_2) \leq 8 < 9 = K(G_1) \cdot K(G_2) \).

**Bibliography**