ON A THEOREM IN COMPLETE \( \mathfrak{a} \)-ADIC RINGS

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Introduction. In this paper we characterize a local ring \( R \) with maximal ideal \( \mathfrak{M} \) and ideal \( \mathfrak{A} \) such that \( R \) is equicharacteristic, \( R/\mathfrak{A} \) is regular, and \( R \) is Hausdorff and complete with respect to the \( \mathfrak{A} \)-adic topology. The key result is an extension of a theorem due to Harrison [5], which is analogous to the first part of Cohen’s Theorem 9 [1]. Methods similar to those used in Geddes [3], [4] and in Curtiss [2] are used.

Definition. A local ring \( R \) with maximal ideal \( \mathfrak{M} \) and an ideal \( \mathfrak{A} \) is called a special local ring if it satisfies the following properties:

1. \( R \) is equicharacteristic.
2. \( R/\mathfrak{A} \) is regular.
3. \( R \) is a complete space with respect to the \( \mathfrak{A} \)-adic topology.

Harrison [5] has proved the following

Theorem A. Let \( R \) be a special local ring. If \( \mathfrak{A}^2 = 0 \), then there exists a subring \( B \) of \( R \) such that \( B + \mathfrak{A} = R \) and \( B \cap \mathfrak{A} = (0) \).

We wish to extend this theorem by showing that the conclusion holds when \( R \) is Hausdorff and \( \mathfrak{A}^2 \neq 0 \). We first prove the following:

Lemma 1. Let \( R \) be a Noetherian ring, containing an identity \( 1 \), \( \mathfrak{A} \) an ideal of \( R \) such that \( R/\mathfrak{A} \) is local, and \( \mathfrak{A}^2 = 0 \). Then \( R \) is local.

Proof. Let \( \mathfrak{M}' = \mathfrak{M}/\mathfrak{A}^2 \), and \( \sigma: R \to R/\mathfrak{A} \) the natural homomorphism. Let \( \mathfrak{M} = \mathfrak{M}/(\mathfrak{M}') \). If \( x \in \mathfrak{M} \), then \( \sigma(x) \) is a unit in \( R/\mathfrak{A} \). Thus, there exists \( y \in R \) such that \( \sigma(x)\sigma(y) = \sigma(1) \), which implies that \( xy - 1 \in \mathfrak{A} \). Let \( xy - 1 = \alpha \), \( \alpha \in \mathfrak{A} \). Then \( x(y - \alpha y) = xy - \alpha xy = 1 + \alpha - \alpha(1 + \alpha) = 1 + \alpha - \alpha - \alpha^2 = 1. \) Hence, \( x \) is a unit in \( R \), and it then follows that \( R \) is local.

Theorem 1. Let \( R \) be a special local ring which is Hausdorff in the \( \mathfrak{A} \)-adic topology. If \( \mathfrak{A}^2 \neq (0) \), then there exists a subring \( B \) of \( R \) such that \( B + \mathfrak{A} = R \), \( B \cap \mathfrak{A} = (0) \).

Proof. Let \( R' = R/\mathfrak{A}^2 \). Then \( R' \) is a local ring with maximal ideal \( \mathfrak{M}' = \mathfrak{M}/\mathfrak{A}^2 \). Moreover, \( R' \) is a special local ring with respect to the ideal \( \mathfrak{A}' = \mathfrak{A}/\mathfrak{A}^2 \). To verify this, we note that (2) in the definition of special local ring follows from \( R'/\mathfrak{A}' = (R/\mathfrak{A})/(\mathfrak{A}/\mathfrak{A}^2) \cong R/\mathfrak{A} \), which is regular. (3) follows from the corresponding property for \( R \). (1) follows from the fact that \( R'/\mathfrak{M}' = (R/\mathfrak{A})/(\mathfrak{M}/\mathfrak{A}^2) \cong R/\mathfrak{M}. \)

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Applying Theorem A to \( R' \), we have a subring \( R'_1 \) of \( R' \) such that \( R'_1 + \mathfrak{A'} = R' \) and \( R'_1 \cap \mathfrak{A'} = \mathfrak{A}^2 \). Letting \( R_1 \) be the inverse image of \( R'_1 \) under the natural homomorphism \( R \to R' = R/\mathfrak{A}^2 \), we have \( R_1 + \mathfrak{A} = R \), and \( R_1 \cap \mathfrak{A} = \mathfrak{A}^2 \). We now assert that \( 1 \in R_1 \). For we have \( 1 = r_1 + a \), where \( r_1 \in R_1 \), \( a \in \mathfrak{A} \). Then \( r_1^2 - r_1 = ar_1 \in R_1 \cap \mathfrak{A} = \mathfrak{A}^2 \). Since \( a^2 \in R_1 \), we have \( 1 - r_1^2 \in R_1 \), and thus \( 1 \in R_1 \).

Now \( (R_1/\mathfrak{A}^4)/(\mathfrak{A}^2/\mathfrak{A}^4) \simeq R_1/\mathfrak{A}^2 \simeq (R_1 \cap \mathfrak{A})/(R_1 \cap \mathfrak{A}) \simeq (R_1 + \mathfrak{A})/\mathfrak{A} \simeq R/\mathfrak{A} \), and hence \( (R_1/\mathfrak{A})/(\mathfrak{A}^2/\mathfrak{A}^4) \) is a local ring and \( (\mathfrak{A}^2/\mathfrak{A}^4)^2 = 0 \). Also, \( 1 \in R_1/\mathfrak{A}^4 \). By Lemma 1, we conclude that \( R_1/\mathfrak{A}^4 \) is a local ring. Let \( R'_1 = R_1/(\mathfrak{A}^2)^2 \), \( \mathfrak{M}' = \mathfrak{A}^2/(\mathfrak{A}^2)^2 \) an ideal in \( R'_1 \), \( \mathfrak{M}' \) the maximal ideal of \( R'_1 \). We now verify that \( R'_1 \) is a special local ring. The residue field of \( R'_1 \) is \( R'_1/\mathfrak{M}' \), and it then follows that \( \text{char } R'_1 = \text{char } R'_1/\mathfrak{M}' \).

Moreover, \( R'_1/\mathfrak{M}' = (R_1/\mathfrak{A})/(\mathfrak{A}^2/\mathfrak{A}^4) \simeq R_1/\mathfrak{A}^2 = R_1/(R_1 \cap \mathfrak{A}) \simeq (R_1 + \mathfrak{A})/\mathfrak{A} = R/\mathfrak{A} \), which implies that \( R'_1/\mathfrak{M}' \) is regular. Also, if \( R \) is complete in its \( \mathfrak{A} \)-topology then it is complete in its \( \mathfrak{A}^2 \)-topology. For if \( \{ a \} \) is a regular sequence in \( R \) with respect to the \( \mathfrak{A} \)-topology, then given any integer \( s \geq 0 \) there exists an integer \( N_1 \) such that if \( n, m \geq N_1 \), then \( a_n - a_m \in \mathfrak{A}^{2s} \). Moreover, \( \{ a \} \) is regular with respect to the \( \mathfrak{A} \)-topology, and thus \( \{ a \} \) has a limit \( a \). Thus, given any \( r \geq 0 \), there exists \( N_2 \) such that if \( m > N_2 \) then \( a_m - a \in \mathfrak{A}^r \). Then if \( n > N = \max(N_1, N_2) \) we have \( a_n - a \in \mathfrak{A}^s + \mathfrak{A}^r \). Since \( \mathfrak{A}^s \) is open in the \( \mathfrak{A} \)-topology, it is closed. Thus, \( \mathfrak{A}^s = \bigcap_{s=1}^{\infty} (\mathfrak{A}^s + \mathfrak{A}^r) \), which implies that \( a_n - a \in \mathfrak{A}^s \) for \( n > N \), and thus \( a \) is the limit of \( \{ a \} \) in the \( \mathfrak{A} \)-topology. Thus, \( R \) is complete in its \( \mathfrak{A} \)-topology. Then \( R_1/\mathfrak{A}^4 \) is complete in the \( \mathfrak{A}^2/\mathfrak{A}^4 \) topology. Then by Theorem A, and going back in the natural homomorphism \( R \to R_1/\mathfrak{A}^4 \), there exists a subring \( R_2 \) of \( R_1 \) such that \( R_2 + \mathfrak{A}^2 = R_1 \) and \( R_2 \cap \mathfrak{A}^2 = \mathfrak{A}^4 = \mathfrak{A}^2 \). Thus, by induction, there exists a sequence of subrings \( \{ R_n \} \), \( R_{n+1} \subseteq R_n \) such that \( R_{n+1} + \mathfrak{A}^s = R_n \) and \( R_{n+1} \cap \mathfrak{A}^s = \mathfrak{A}^{s+1} \). Let \( B = \bigcap_{i=1}^{\infty} R_i \). We then have \( R_i \cap \mathfrak{A} = \mathfrak{A}^s \), for \( \mathfrak{A}^s = R_i \cap \mathfrak{A}^{s-1} = R_i \cap R_{i-1} \cap \mathfrak{A}^{s-2} = R_i \cap R_{i-1} \cap \mathfrak{A} \cdots \cap R_1 \cap \mathfrak{A} = R_i \cap \mathfrak{A} \).

Next, we show that \( B + \mathfrak{A} = R \). We first have that \( R = R_1 + \mathfrak{A} \), and by induction get \( R = R_n + \mathfrak{A} \), for any \( n \). Thus, if \( a \in R \), then for any \( n \) there exists \( b_n \in R_n \) such that \( a = b_n(\mathfrak{A}) \). Then \( b_n - b_{n+1} \in R_n \cap \mathfrak{A} \).

Since \( R_n \cap \mathfrak{A} = \mathfrak{A}^s \), \( \{ b_n \} \) is a regular sequence in \( R \). Since \( R \) is complete, \( \{ b_n \} \) has a limit \( b \) in \( R \). Also, \( R_n \subseteq \mathfrak{A}^s \), and hence \( R_n \) is closed in the \( \mathfrak{A} \)-adic topology. Thus, \( b \in R_n \) for all \( n \), which implies that
Let $R$ be a Noetherian ring which is complete with respect to the $\mathfrak{A}$-adic topology, where $\mathfrak{A} = (u_1, u_2, \ldots, u_n)$. Let $S = \mathbb{R}[x_1, x_2, \ldots, x_m]$ be the ring of formal power series in $m$ indeterminates over $R$. Then $S$ is a Noetherian ring which is complete with respect to the $\mathfrak{A}^*$-adic topology, where $\mathfrak{A}^* = (u_1, u_2, \ldots, u_n, x_1, x_2, \ldots, x_m)$. Moreover, if the above basis for $\mathfrak{A}$ is minimal then so is the indicated basis for $\mathfrak{A}^*$.

Proof. To show that if $R$ is Noetherian then $S$ is Noetherian we proceed as in the Hilbert basis theorem. We now show that $S$ is complete in the $\mathfrak{A}^*$-adic topology. Let $\{f^i\}$ be a regular sequence in $S$. Then $f^i = \sum_{k=0}^{\infty} f_k^i$, where $f_k^i$ is a form in $x_1, x_2, \ldots, x_m$ of degree $k$. Since $\{f^i\}$ is regular in $S$, given any $s > 0$ then $f^i - f^{i+1} \in \mathfrak{A}^{s-*}$ for $i$ sufficiently large. Now

$$\mathfrak{A}^{s-*} = \left\{ \sum_{k=0}^{\infty} t_k | t_k \in \mathbb{R}^{s-k} \cdot \mathbb{R}[x_1, x_2, \ldots, x_n] \forall k < s \right\}.$$  

Fixing $k$, and for $s > k$, we then have that the coefficients of $f_k^i - f_k^{i+1} \in \mathfrak{A}^{s-k}$. Thus, the coefficients of each monomial in $f_k^i$ form a regular sequence in $R$, and since $R$ is complete they have a limit in $R$. Thus there exists a form $f_k$ of degree $k$ such that $\lim_{i \to \infty} f_k^i = f_k$. Let $f = \sum f_k$. Then $f^i - f = \sum_k (f_k^i - f_k)$, and for any $s \geq 0$, $f_k^i - f_k \in \mathfrak{A}^{s-k} \cdot \mathbb{R}[x_1, x_2, \ldots, x_n]$ for all $k < s$ and $i$ sufficiently large. Hence, $f^i - f \in \mathfrak{A}^{s-*}$ for $i$ sufficiently large, which implies that $S$ is complete.

Now suppose that the above basis of $\mathfrak{A}$ is minimal and that the basis of $\mathfrak{A}^*$ is not minimal. Thus, suppose $x_1 = \sum_{i=1}^{n} f_i u_i + \sum_{j=2}^{m} g_j x_j$, $f_i, g_j \in S$. We expand the right-hand side into a power series in $x_1, x_2, \ldots, x_n$ and observe that $\sum_{j=2}^{m} g_j x_j$ contributes no term in $x_1$ alone. Thus, if $\sum_{i=1}^{n} f_i u_i$ contributes no such term then $1 = 0$. If $\sum_{i=1}^{n} f_i u_i$ does contribute such a term then its coefficient $\in \mathfrak{A}$, which implies that $1 \in \mathfrak{A}$ and thus $\mathfrak{A} = \mathbb{R}$, a contradiction. If we assume that $u_1 = \sum_{i=2}^{n} f_i u_i + \sum_{j=1}^{m} g_j x_j$, $f_i, g_j \in S$, then $u_1 = \sum_{i=2}^{n} f_i u_i$, where $f_i$ is the constant term of $f_i$, which contradicts the minimality of the basis $(u_1, u_2, \ldots, u_n)$. This completes the proof.

The following theorem characterizes a special local ring which is Hausdorff with respect to the $\mathfrak{A}$-adic topology, and is analogous to the second part of Cohen's Theorem 9 [1]. The proof follows along the lines of Cohen's proof.
Theorem 2. Let $R$ be a special local ring with maximal ideal $\mathfrak{M}$, Hausdorff in the $\mathfrak{M}$-adic topology, where $\mathfrak{M}$ is an ideal having a maximal basis of $n$ elements. Then $R$ is a homomorphic image of a formal power series ring $(R/\mathfrak{M}) \{x_1, x_2, \ldots, x_n\}$.

Proof. By Theorem 1, there exists a subring $B$ of $R$ such that $B + \mathfrak{M} = R$, $B \cap \mathfrak{M} = (0)$. Let $\mathfrak{M} = (u_1, u_2, \ldots, u_n)$. We first show that $R$ is the closure of $B[u_1, u_2, \ldots, u_n]$. Let $c \in R$. Then we find a sequence $\{c_k\}$, $c_k \in B[u_1, u_2, \ldots, u_n]$, such that $\lim c_k = c$, i.e., we show that $c - c_k \in \mathfrak{M}^{k+1}$ by induction.

Since $c \in R$, there exists $b \in B$ such that $c = b(\mathfrak{M})$. Let $c_0 = b$. Assume that $c_k$ has been defined. Then $c - c_k \in \mathfrak{M}^{k+1}$, which implies that $c - c_k = \sum_{i=1}^{n} r_i v_i$, where $r_i \in R$, and $v_i$ is a power product of $u_1, u_2, \ldots, u_n$ of degree $k+1$. Since $r_i \in R$, there exists $b_i \in R$ such that $r_i = b_i(b_i)$. Then $c - c_k \equiv \sum_{i=1}^{n} b_i(b_i) \mathfrak{M}^{k+1}$. Letting $c_{k+1} = c_k + \sum_{i=1}^{n} b_i b_i$, we have constructed the desired sequence $\{c_k\}$. Now in the natural mapping of $R$ onto $R/\mathfrak{M}$, $B$ is mapped onto $R/\mathfrak{M}$. There exists a unique homomorphism $T$ of $(R/\mathfrak{M}) \{x_1, x_2, \ldots, x_n\}$ onto $B[u_1, u_2, \ldots, u_n]$ such that $x_i T = u_i$ and $T$ restricted to $R/\mathfrak{M}$ is the inverse of the natural mapping of $B$ onto $R/\mathfrak{M}$. $T$ is continuous, where the topology of $(R/\mathfrak{M}) \{x_1, x_2, \ldots, x_n\}$ is the relative topology determined by $(R/\mathfrak{M}) \{x_1, x_2, \ldots, x_n\}$. Since $R$ is complete and is the closure of $B[u_1, u_2, \ldots, u_n]$ and $(R/\mathfrak{M}) \{x_1, x_2, \ldots, x_n\}$ is the closure of $(R/\mathfrak{M}) \{x_1, x_2, \ldots, x_n\}$, then $T$ can be uniquely extended to a homomorphism of $(R/\mathfrak{M}) \{x_1, x_2, \ldots, x_n\}$ onto $R$. This completes the proof of Theorem 2.

Bibliography


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