ON A THEOREM IN COMPLETE $\mathfrak{A}$-ADIC RINGS

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Introduction. In this paper we characterize a local ring R with maximal ideal $\mathfrak{M}$ and ideal $\mathfrak{A}$ such that R is equicharacteristic, $R/\mathfrak{A}$ is regular, and R is Hausdorff and complete with respect to the $\mathfrak{A}$-adic topology. The key result is an extension of a theorem due to Harrison [5], which is analogous to the first part of Cohen's Theorem 9 [1]. Methods similar to those used in Geddes [3], [4] and in Curtiss [2] are used.

DEFINITION. A local ring R with maximal ideal $\mathfrak{M}$ and an ideal $\mathfrak{A}$ is called a special local ring if it satisfies the following properties:

(1) R is equicharacteristic.

(2) $R/\mathfrak{A}$ is regular.

(3) R is a complete space with respect to the $\mathfrak{A}$-adic topology.

Harrison [5] has proved the following

THEOREM A. *Let R be a special local ring. If $\mathfrak{A}^2=0$, then there exists a subring B of R such that $B+\mathfrak{A}=R$ and $B\cap\mathfrak{A}=(0)$.*

We wish to extend this theorem by showing that the conclusion holds when R is Hausdorff and $\mathfrak{A}^2\neq 0$. We first prove the following:

LEMMA 1. *Let R be a Noetherian ring, containing an identity 1, $\mathfrak{A}$ an ideal of R such that $R/\mathfrak{A}$ is local, and $\mathfrak{A}^2=0$. Then R is local.*

PROOF. Let $\mathfrak{M}'$ be the unique maximal ideal of $R/\mathfrak{A}$, and $\sigma: R\to R/\mathfrak{A}$ the natural homomorphism. Let $\mathfrak{M}=\mathfrak{M}^{-1}(\mathfrak{M}')$. If $x\notin\mathfrak{M}$, then $\sigma(x)$ is a unit in $R/\mathfrak{A}$. Thus, there exists $y\in R$ such that $\sigma(x)\sigma(y)=\sigma(1)$, which implies that $xy-1\in\mathfrak{A}$. Let $xy-1=\alpha$, $\alpha\in\mathfrak{A}$. Then $x(y-\alpha y)=xy-\alpha xy=1+\alpha-\alpha(1+\alpha)=1+\alpha-\alpha-\alpha^2=1$. Hence, x is a unit in R, and it then follows that R is local.

THEOREM 1. *Let R be a special local ring which is Hausdorff in the $\mathfrak{A}$-adic topology. If $\mathfrak{A}^2\neq(0)$, then there exists a subring B of R such that $B+\mathfrak{A}=R$, $B\cap\mathfrak{A}=(0)$.*

PROOF. Let $R'=R/\mathfrak{A}^2$. Then R' is a local ring with maximal ideal $\mathfrak{M}'=\mathfrak{M}/\mathfrak{A}^2$. Moreover, R' is a special local ring with respect to the ideal $\mathfrak{A}'=\mathfrak{A}/\mathfrak{A}^2$. To verify this, we note that (2) in the definition of special local ring follows from $R'/\mathfrak{A}'=(R/\mathfrak{A}^2)/(\mathfrak{A}/\mathfrak{A}^2)\simeq R/\mathfrak{A}$, which is regular. (3) follows from the corresponding property for R. (1) follows from the fact that $R'/\mathfrak{M}'=(R/\mathfrak{A}^2)/(\mathfrak{M}/\mathfrak{A}^2)\simeq R/\mathfrak{M}$.

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Applying Theorem A to R', we have a subring R_1' of R' such that $R_1'+\mathfrak{A}'=R'$ and $R_1'\cap\mathfrak{A}'=\mathfrak{A}'^2$. Letting R_1 be the inverse image of R_1' under the natural homomorphism $R\to R'=R/\mathfrak{A}^2$, we have $R_1+\mathfrak{A}=R$, and $R_1\cap\mathfrak{A}=\mathfrak{A}^2$. We now assert that $1\in R_1$. For we have $1=r_1+a$, where $r_1\in R_1$, $a\in\mathfrak{A}$. Then $r_1^2-r_1=ar_1\in R_1\cap\mathfrak{A}=\mathfrak{A}^2$. Since $a^2\in R_1$, we have $1-r_1^2\in R_1$, and thus $1\in R_1$.

Now $(R_1/\mathfrak{A}^4)/(\mathfrak{A}^2/\mathfrak{A}^4)\simeq R_1/\mathfrak{A}^2\simeq R_1/(R_1\cap\mathfrak{A})\simeq(R_1+\mathfrak{A})/\mathfrak{A}\simeq R/\mathfrak{A}$, and hence $(R_1/\mathfrak{A}^4)/(\mathfrak{A}^2/\mathfrak{A}^4)$ is a local ring and $(\mathfrak{A}^2/\mathfrak{A}^4)^2=\bar{0}$. Also, $1\in R_1/\mathfrak{A}^4$. By Lemma 1, we conclude that $R_1/\mathfrak{A}^4$ is a local ring. Let $R_1''=R_1/(\mathfrak{A}^2)^2$, $\mathfrak{A}''=\mathfrak{A}^2/(\mathfrak{A}^2)^2$ an ideal in R_1'', $\mathfrak{M}''$ the maximal ideal of R_1''. We now verify that R_1'' is a special local ring. The residue field of R_1'' is $R_1''/\mathfrak{M}''$, and it then follows that char $R_1''=$ char $R_1''/\mathfrak{M}''$. Moreover, $R_1''/\mathfrak{A}''=(R_1/\mathfrak{A}^4)/(\mathfrak{A}^2/\mathfrak{A}^4)\simeq R_1/\mathfrak{A}^2=R_1/(R_1\cap\mathfrak{A})\simeq(R_1+\mathfrak{A})/\mathfrak{A}=R/\mathfrak{A}$, which implies that $R_1''/\mathfrak{A}''$ is regular. Also, if R is complete in its $\mathfrak{A}$-topology then it is complete in its $\mathfrak{A}^2$-topology. For if $\{a\}$ is a regular sequence in R with respect to the $\mathfrak{A}^2$-topology, then given any integer $s\geqq 0$ there exists an integer N_1 such that if n, $m\geqq N_1$ then $a_n-a_m\in\mathfrak{A}^{2s}$. Moreover, $\{a_j\}$ is regular with respect to the $\mathfrak{A}$-topology, and thus $\{a_j\}$ has a limit a. Thus, given any $r\geqq 0$, there exists N_2 such that if $m>N_2$ then $a_m-a\in\mathfrak{A}^r$. Then if $n>N=\max(N_1, N_2)$ we have $a_n-a\in\mathfrak{A}^{2s}+\mathfrak{A}^r$. Since $\mathfrak{A}^{2s}$ is open in the $\mathfrak{A}$-topology, it is closed. Thus, $\mathfrak{A}^{2s}=\bigcap_{r=1}^{\infty}(\mathfrak{A}^{2s}+\mathfrak{A}^r)$, which implies that $a_n-a\in\mathfrak{A}^{2s}$ for $n>N$, and thus a is the limit of $\{a_j\}$ in the $\mathfrak{A}^2$-topology. Thus, R is complete in its $\mathfrak{A}^2$ topology. Since $R_1\cap\mathfrak{A}^s=\mathfrak{A}^s$, $s\geqq 2$, then R_1 is complete with respect to the $\mathfrak{A}^2$-adic topology. Then $R_1/\mathfrak{A}^4$ is complete in the $\mathfrak{A}^2/\mathfrak{A}^4$ topology. Then by Theorem A, and going back in the natural homomorphism $R_1\to R_1/\mathfrak{A}^4$, there exists a subring R_2 of R_1 such that $R_2+\mathfrak{A}^2=R_1$ and $R_2\cap\mathfrak{A}^2=\mathfrak{A}^4$. Thus, by induction, there exists a sequence of subrings $\{R_n\}$, $R_{n+1}\subset R_n$ such that $R_{n+1}+\mathfrak{A}^{2^n}=R_n$ and $R_{n+1}\cap\mathfrak{A}^{2^n}=\mathfrak{A}^{2^{n+1}}$. Let $B=\bigcap_{i=1}^{\infty}R_i$. We then have $R_i\cap\mathfrak{A}=\mathfrak{A}^{2^i}$, for $\mathfrak{A}^{2^i}=R_i\cap\mathfrak{A}^{2^{i-1}}=R_i\cap R_{i-1}\cap\mathfrak{A}^{2^{i-2}}=R_i\cap R_{i-1}\cap R_{i-2}\cap\cdots\cap R_1\cap\mathfrak{A}=R_i\cap\mathfrak{A}$. Then

$$B\cap\mathfrak{A}=\left(\bigcap_{i=1}^{\infty}R_i\right)\cap\mathfrak{A}=\bigcap_{i=1}^{\infty}(R_i\cap\mathfrak{A})=\bigcap_{i=1}^{\infty}\mathfrak{A}^{2^i}=(0).$$

Next, we show that $B+\mathfrak{A}=R$. We first have that $R=R_1+\mathfrak{A}$, and by induction get $R=R_n+\mathfrak{A}$, for any n. Thus, if $a\in R$, then for any n there exists $b_n\in R_n$ such that $a\equiv b_n(\mathfrak{A})$. Then $b_n-b_{n+1}\in R_n\cap\mathfrak{A}$. Since $R_n\cap\mathfrak{A}=\mathfrak{A}^{2^n}$, $\{b_n\}$ is a regular sequence in R. Since R is complete, $\{b_n\}$ has a limit b in R. Also, $R_n\supset\mathfrak{A}^{2^n}$, and hence R_n is closed in the $\mathfrak{A}$-adic topology. Thus, $b\in R_n$ for all n, which implies that

$b \in B$. Moreover, since $a \equiv b_n(\mathfrak{A})$, $b_n \to b$ and we have $a \equiv b(\mathfrak{A})$. Thus, $B + \mathfrak{A} = R$. This completes the proof of Theorem 1.

LEMMA 2. *Let R be a Noetherian ring which is complete with respect to the $\mathfrak{A}$-adic topology, where $\mathfrak{A} = (u_1, u_2, \cdots, u_n)$. Let $S = R\{x_1, x_2, \cdots, x_m\}$ be the ring of formal power series in m indeterminates over R. Then S is a Noetherian ring which is complete with respect to the $\mathfrak{A}^*$-adic topology, where $\mathfrak{A}^* = (u_1, u_2, \cdots, u_n, x_1, x_2, \cdots, x_m)$. Moreover, if the above basis for $\mathfrak{A}$ is minimal then so is the indicated basis for $\mathfrak{A}^*$.*

PROOF. To show that if R is Noetherian then S is Noetherian we proceed as in the Hilbert basis theorem. We now show that S is complete in the $\mathfrak{A}^*$-adic topology. Let $\{f^i\}$ be a regular sequence in S. Then $f^i = \sum_{k=0}^{\infty} f_k^i$, where f_k^i is a form in $x_1, x_2, \cdots, x_m$ of degree k. Since $\{f^i\}$ is regular in S, given any $s > 0$ then $f^i - f^{i+1} \in \mathfrak{A}^{*s}$ for i sufficiently large. Now

$$\mathfrak{A}^{*s} = \left\{ \sum_{k=0}^{\infty} t_k \mid t_k \in \mathfrak{A}^{s-k} \cdot R[x_1, x_2, \cdots, x_n] \ \forall\ k < s \right\}.$$

Fixing k, and for $s > k$, we then have that the coefficients of $f_k^i - f_k^{i+1} \in \mathfrak{A}^{s-k}$. Thus, the coefficients of each monomial in f_k^i form a regular sequence in R, and since R is complete they have a limit in R. Thus there exists a form f_k of degree k such that $\lim_{i\to\infty} f_k^i = f_k$. Let $f = \sum_k f_k$. Then $f^i - f = \sum_k (f_k^i - f_k)$, and for any $s \geqq 0$, $f_k^i - f_k \in \mathfrak{A}^s \cdot R[x_1, x_2, \cdots, x_n]$ for all $k < s$ and i sufficiently large. Hence, $f^i - f \in \mathfrak{A}^{*s}$ for i sufficiently large, which implies that S is complete.

Now suppose that the above basis of $\mathfrak{A}$ is minimal and that the basis of $\mathfrak{A}^*$ is not minimal. Thus, suppose $x_1 = \sum_{i=1}^n f_i u_i + \sum_{j=2}^m g_j x_j$, $f_i, g_j \in S$. We expand the right-hand side into a power series in $x_1, x_2, \cdots, x_m$ and observe that $\sum_{j=2}^m g_j x_j$ contributes no term in x_1 alone. Thus, if $\sum_{i=1}^n f_i u_i$ contributes no such term then $1 = 0$. If $\sum_{i=1}^n f_i u_i$ does contribute such a term then its coefficient $\in \mathfrak{A}$, which implies that $1 \in \mathfrak{A}$ and thus $\mathfrak{A} = R$, a contradiction. If we assume that $u_1 = \sum_{i=2}^n f_i' u_i + \sum_{j=1}^m g_j x_j$, f_i, $g_j \in S$, then $u_1 = \sum_{i=2}^n f_i^n u_i$, where f_i' is the constant term of f_i, which contradicts the minimality of the basis $(u_1, u_2, \cdots, u_n)$. This completes the proof.

The following theorem characterizes a special local ring which is Hausdorff with respect to the $\mathfrak{A}$-adic topology, and is analogous to the second part of Cohen's Theorem 9 [1]. The proof follows along the lines of Cohen's proof.

THEOREM 2. *Let R be a special local ring with maximal ideal $\mathfrak{M}$, Hausdorff in the $\mathfrak{A}$-adic topology, where $\mathfrak{A}$ is an ideal having a maximal basis of n elements. Then R is a homomorphic image of a formal power series ring* $(R/\mathfrak{A})\{x_1, x_2, \cdots, x_n\}$.

PROOF. By Theorem 1, there exists a subring B of R such that $B+\mathfrak{A}=R$, $B\cap\mathfrak{A}=(0)$. Let $\mathfrak{A}=(u_1, u_2, \cdots, u_n)$. We first show that R is the closure of $B[u_1, u_2, \cdots, u_n]$. Let $c\in R$. Then we find a sequence $\{c_k\}$, $c_k\in B[u_1, u_2, \cdots, u_n]$, such that $\lim c_k=c$, i.e., we show that $c-c_k\in\mathfrak{A}^{k+1}$ by induction.

Since $c\in R$, there exists $b\in B$ such that $c\equiv b(\mathfrak{A})$. Let $c_0=b$. Assume that c_k has been defined. Then $c-c_k\in\mathfrak{A}^{k+1}$, which implies that $c-c^k=\sum_{i=1}^n r_iv_i$, where $r_i\in R$, and v_i is a power product of $u_1, u_2, \cdots, u_n$ of degree $k+1$. Since $r_i\in R$, there exists $b_i\in R$ such that $r_i\equiv b_i(\mathfrak{A})$. Then $c-c_k\equiv\sum_{i=1}^n b_iv_i(\mathfrak{A}^{k+2})$. Letting $c_{k+1}=c_k+\sum_{i=1}^n b_iv_i$, we have constructed the desired sequence $\{c_k\}$. Now in the natural mapping of R onto $R/\mathfrak{A}$, B is mapped onto $R/\mathfrak{A}$. There exists a unique homomorphism T of $(R/\mathfrak{A})[x_1, x_2, \cdots, x_n]$ onto $B[u_1, u_2, \cdots, u_n]$ such that $x_iT=u_i$ and T restricted to $R/\mathfrak{A}$ is the inverse of the natural mapping of B onto $R/\mathfrak{A}$. T is continuous, where the topology of $(R/\mathfrak{A})[x_1, x_2, \cdots, x_n]$ is the relative topology determined by $(R/\mathfrak{A})\{x_1, x_2, \cdots, x_n\}$. Since R is complete and is the closure of $B[u_1, u_2, \cdots, u_n]$ and $(R/\mathfrak{A})\{x_1, x_2, \cdots, x_n\}$ is the closure of $(R/\mathfrak{A})[x_1, x_2, \cdots, x_n]$, then T can be uniquely extended to a homomorphism of $(R/\mathfrak{A})\{x_1, x_2, \cdots, x_n\}$ onto R. This completes the proof of Theorem 2.

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