ON AN INEQUALITY CONSIDERED BY ROBERTSON

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Let $S$ denote the usual family of normalized univalent functions $f(z)$ in the unit circle whose power series expansion about the origin is given by

$$f(z) = z + \sum_{n=2}^{\infty} A_n z^n$$

where we write also for convenience $A_1 = 1$. Robertson [2] has recently considered an inequality involving the coefficients of (1) which if valid for the family $S$ would imply the truth of the Bieberbach conjecture. He verified that this inequality

$$|m|A_m| - n|A_n| \leq m^2 - n^2, \quad m, n = 1, 2, \ldots,$$

did hold for several special subclasses of $S$. He made no conjecture concerning its general validity for $S$. Indeed, it is not in general valid and in particular fails for $m = 3, n = 2$. In this case the inequality would be $|3|A_3| - 2|A_2| \leq 5.$ Our theorem will be stated in the following form.

**Theorem.** For the coefficients of the function (1) there hold the inequalities

$$-2 \leq 3|A_3| - 2|A_2| \leq \frac{1}{2} \tau_0 + \frac{3}{8} \tau_0^2.$$

The lower bound is attained only for the functions $z(1 + e^{i\phi}z + e^{2i\phi}z^2)^{-1}$, $\phi$ real. The upper bound is attained only for the functions $[f(z^{-1}, \tau_0, \phi)]^{-1}$ as defined in [1, p. 171] with $\phi$ real and $\tau_0$ the larger root of the equation

$$\frac{3}{2} \tau \log(\tau/4) + 1 = 0$$

for $\tau > 0$. This upper bound is in particular greater than 5.

The lower bound is elementary. In particular $3|A_3| - 2|A_2| > -2$ if $|A_2| < 1$ while for $1 \leq |A_2| \leq 2$ we have from the familiar result $|A_3 - A_2|^2 \leq 1$ the inequality

$$3|A_3| - 2|A_2| \geq 3|A_3| - 3|A_2|^2 + 3|A_2|^2 - 2|A_2|^2 

\geq -3 + \min_{1 \leq x \leq 2} (3x^2 - 2x) \geq -2.$$
Equality is possible only if both $|A_2| = 1$ and $|A_3 - A_2^2| = 1$, which requires a function of the form prescribed in our theorem.

For the upper bound we recall the result of Corollary 5 in [1] by which if $|A_2| = 0$, $|A_2| \leq 1$ while if

$$|A_2| \leq \frac{1}{2} \tau(1 - \log(\tau/4))$$

with $0 < \tau \leq 4$, we have

$$|A_3| \leq 1 + \frac{1}{8} \tau^2 - \frac{1}{4} \tau^2 \log(\tau/4) + \frac{1}{4} \tau^2 (\log(\tau/4))^2.$$

The first eventuality evidently does not correspond to the maximum and we have $3|A_3| - 2|A_2| \leq \max_{0 \leq \tau \leq 4} \Psi(\tau)$ where

$$\Psi(\tau) = 3, \quad \tau = 0,$$

$$= 3 + \frac{3}{8} \tau^2 - \frac{3}{4} \tau^2 \log(\tau/4) + \frac{3}{4} \tau^2 (\log(\tau/4))^2 - \tau(1 - \log(\tau/4)), \quad 0 < \tau \leq 4.$$

A direct calculation gives for $0 < \tau \leq 4$

$$d\Psi(\tau)/d\tau = \log(\tau/4)((3/2)\tau \log(\tau/4) + 1).$$

To find the maximum of $\Psi(\tau)$ we observe the function $F(\tau) = (3/2)\tau \log(\tau/4) + 1$. Its derivative with respect to $\tau$ is $(3/2)(\log(\tau/4) + 1)$. Thus, on the interval $[0, 4]$, $F(\tau)$ starts with the value 1 at $\tau = 0$ (defined by continuity), decreases to the point $\tau = 4e^{-1}$ at which $F(\tau)$ is negative, then increases to the value 1 at $\tau = 4$. In particular, $F(\tau)$ has two zeros $\tau_0'$, $\tau_0$ on $(0, 4)$, $\tau_0' < \tau_0$. Correspondingly, $\Psi(\tau)$ starts with the value 3 at $\tau = 0$, decreases to the point $\tau = \tau_0'$, increases to the point $\tau = \tau_0$, then decreases to the value 5 at $\tau = 4$. The maximum of $\Psi(\tau)$ on $[0, 4]$ evidently does not occur for $\tau = 0$, thus it does occur for $\tau = \tau_0$, and this maximum exceeds 5. One could readily find an explicit numerical approximation for $\Psi(\tau_0) = 8/3 - 1/2 \tau_0 + (3/2) \tau_0^2$. The equality statement of our theorem follows at once from the corresponding statement in [1, Corollary 5].

Since the functions used to provide this counterexample to the validity of Robertson's inequality for the family $S$ are those used to prove $|A_3| \leq 3$ in [1], this situation seems to shed little light on the validity of the Bieberbach conjecture.

Bibliography


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