A COMPATIBILITY THEOREM FOR TWO
POINT BOUNDARY PROBLEMS

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Let $P$ be an $n$ by $n$ matrix whose elements are complex valued functions continuous in $(t, \lambda)$ for $t$ in the closed interval $[a, b]$ and $\lambda$ in the closed disk $R = \{ \lambda : |\lambda - \lambda_0| \leq r, r > 0 \}$ in the complex plane. Consider the $n$-vector differential equation

$$
\frac{dy}{dt} = Py
$$

together with the homogeneous $n$-vector two point boundary conditions

$$
Uy \equiv Ay(a) + By(b) = 0,
$$

where $A$ and $B$ are $n$ by $n$ matrices the elements of which are continuous complex valued functions on $R$.

The compatibility of (1), (2) at $\lambda \in R$ is defined to be the maximum number of linearly independent solutions of (1), (2) corresponding to $\lambda$. Let $y_1, \cdots, y_n$ be a fundamental set of solutions of (1) continuous in $(t, \lambda)$ for $t \in [a, b]$ and $\lambda \in R$, hence uniformly continuous there. A necessary and sufficient condition for the compatibility of (1), (2) at $\lambda$ to be $k$ is that the rank of the $n$ by $n$ matrix $V(\lambda)$ with elements $U_{ji}(\lambda)$ be of rank $n - k$. It is known [1], [2] that if the compatibility of (1), (2) is constant in some neighborhood of $\lambda_0$ and $x(t)$ is a solution of (1), (2) for $\lambda = \lambda_0$, then there exists a solution $x(t, \lambda)$ of (1), (2) which is uniformly continuous in $(t, \lambda)$ for $t \in [a, b]$ and $\lambda$ in some neighborhood of $\lambda_0$ and which is such that $x(t, \lambda_0) = x(t)$ so that $x(t, \lambda) \rightarrow x(t)$ uniformly on $[a, b]$. The question, a natural one, as to what can be said in case the compatibility is not constant in any neighborhood of $\lambda_0$ was brought to the attention of the author by W. M. Whyburn [3]. An answer to the question is in the following theorem.

**Theorem 1.** If the compatibility of (1), (2) at $\lambda_0$ is $k_0 > 0$ and is not identically zero in any deleted neighborhood of $\lambda_0$, then there exists an integer $k$ with $0 < k \leq k_0$, an infinite sequence $\{ \lambda_m \}$ of distinct values of $\lambda$ converging to $\lambda_0$ at each $\lambda_m$ of which the compatibility of (1), (2) is $k$, and correspondingly $k$ linearly independent sequences $\{ x_1(t, \lambda_m) \}$, $\cdots$, $\{ x_k(t, \lambda_m) \}$ of solutions of (1), (2) for $\lambda = \lambda_m$ which converge uni-

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formly on \([a, b]\) to some \(k\) linearly independent solutions \(x_1(t), \ldots, x_k(t)\) of (1), (2) for \(\lambda = \lambda_0\).

Proof. \(V(\lambda)\) as defined above is continuous on \(R\), implying the compatibility of (1), (2) is less than or equal to \(k_0\) at each \(\lambda\) in some neighborhood of \(\lambda_0\). There exists an integer \(k\), \(0 < k \leq k_0\), and an infinite sequence \(S\) of distinct values of \(\lambda\) converging to \(\lambda_0\) at each term of which the compatibility of (1), (2) is \(k\). The case \(k = n\) is disposed of as being trivial since in this case the rank of \(V(\lambda)\) is zero for \(\lambda = \lambda_0\) and every \(\lambda \in S\), implying \(y_1, \ldots, y_n\) are linearly independent solutions of (1), (2) for \(\lambda = \lambda_0\) and every \(\lambda \in S\). For the case \(k < n\) then, since \(V(\lambda)\) is \(n\) by \(n\) with \(n\) finite, it is possible to select an infinite subsequence \(\{\lambda_m\}\) of \(S\) together with two finite subsequences \(i_1, \ldots, i_{n-k}\) and \(j_1, \ldots, j_{n-k}\) of the finite sequence \(1, \ldots, n\) which for every \(m\) satisfies (i) the \(n-k\) by \(n-k\) matrix \(\delta_m = [U_{ij}y_{ij}(\lambda_m)]\) is such that \(\det \delta_m \neq 0\) and (ii) if \(h_1, \ldots, h_{n-k}\) is any subsequence of \(1, \ldots, n\), then the \(n-k\) by \(n-k\) matrix \(\Delta_m = [U_{ij}y_{ij}(\lambda_m)]\) is such that \(|\det \Delta_m| \leq |\det \delta_m|\) and, moreover, \(\lim \det \Delta_m/\det \delta_m\) exists, in fact is bounded by one. As a matter of convenience it is assumed that \(i_1 = j_1 = 1, \ldots, i_{n-k} = j_{n-k} = n-k\) so that \(\delta_m = [U_{ij} y_{ij}(\lambda_m)]\), \(i, j = 1, \ldots, n-k\). Now, with \(y_j = (y_{ij}, \ldots, y_{nj})\), consider the \(k\) infinite sequences of vectors \(\{x_j(t, \lambda_m)\}\), \(j = 1, \ldots, k\), where for each \(j\) the \(n\) components \(x_{ij}, \ldots, x_{nj}\) of \(x_j\) are determinants of order \(n-k+1\) defined by

\[
x_{ij}(t, \lambda_m) = \begin{vmatrix} y_{i1}(t, \lambda_m) & \cdots & y_{in-k}(t, \lambda_m) & c_m y_{in-k+j}(t, \lambda_m) \\ U_{1y1}(\lambda_m) & \cdots & U_{1yn-k}(\lambda_m) & c_m U_{1y_{n-k+j}}(\lambda_m) \\ U_{n-ky1}(\lambda_m) & \cdots & U_{n-ky_{n-k}}(\lambda_m) & c_m U_{n-ky_{n-k+j}}(\lambda_m) \\ \end{vmatrix}
\]

with \(c_m = (-1)^{n-k}/\det \delta_m\). Clearly, \(x_1(t, \lambda_m), \ldots, x_k(t, \lambda_m)\) are linearly independent solutions of (1), (2) for \(\lambda = \lambda_m\) inasmuch as each \(x_j\) is a linear combination of \(y_1, \ldots, y_{n-k}, y_{n-k+j}\) with the coefficient of \(y_{n-k+j}\) equal to one. Moreover, \(\lim x_j(t, \lambda_m)\) exists; indeed, because

\[
x_j(t) = \lim x_j(t, \lambda_m)
\]

for some \(\gamma_{j1}, \ldots, \gamma_{jn-k}\), it follows that \(x_1(t), \ldots, x_k(t)\) are linearly independent solutions of (1), (2) for \(\lambda = \lambda_0\) and that the convergence is uniform on \([a, b]\), completing the proof.

Let \(p_1, \ldots, p_n\) be complex valued functions continuous in \((t, \lambda)\) for \(t \in [a, b]\) and \(\lambda \in R\). Theorem 2 below concerns itself with the \(n\)th order linear scalar differential equation
(3) \[ y^{(n)} + p_1 y^{(n-1)} + \cdots + p_n y = 0 \]
subject to (2). Its proof reflects a simple application of Theorem 1.

**Theorem 2.** Suppose that \( \int_a^b u(t)v(t)\,dt = 0 \) for every pair of solutions \( u, v \) of (3), (2) corresponding to distinct values of \( \lambda \). Then there exists a deleted neighborhood of \( \lambda_0 \) at each \( \lambda \) of which the compatibility of (3), (2) is zero.

**Proof.** If the theorem is false, Theorem 1 then guarantees the existence of an infinite sequence \( \{\lambda_m\} \) of distinct values of \( \lambda \) converging to \( \lambda_0 \) and a sequence \( \{x_m\} \) of solutions of (3), (2) for \( \lambda = \lambda_m \) converging uniformly on \([a, b]\) to \( x \), a solution of (3), (2) for \( \lambda = \lambda_0 \) which is not the trivial solution. But \( \int_a^b x x_m = 0 \) for every \( m \) together with uniform convergence implies \( \int_a^b x x = 0 \), an impossibility.

**Corollary.** The eigenvalues of a regular selfadjoint problem on a finite interval are isolated.

**References**


3. ———, *Differential systems with boundary conditions at more than two points*, Proc. Conference Differential Equations, Univ. of Maryland Bookstore, College Park, Md., 1956, pp. 1–21.

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