ON A CLASS OF PERTURBATIONS OF THE HARMONIC OSCILLATOR

PHILIP HARTMAN

1. The following theorem, concerning solutions of

\[ y'' + [1 + f(x) + h(x) \cos 2\eta x]y = 0, \]

was proved by Atkinson in the cases \( \alpha = 1 \) (where the sum in (1.6) is empty) and \( \alpha = 2 \); see [1, p. 349 and p. 355]. In [3], Kelman and Madsen formulated the general result (\( \alpha = 1, 2, \ldots \)) and proved it using different methods.

**Theorem 1.1** [3]. Let \( f(x) \in L^1[0, \infty) \); \( h(x) \) of bounded variation on \([0, \infty)\) for which there exists an integer \( \alpha > 0 \) satisfying

\[
\int_0^\infty |h|^\alpha dx = \infty \quad \text{and} \quad \int_0^\infty |h|^{\alpha+1} dx < \infty;
\]

\( \eta > 0 \) a constant satisfying

\( 0 < \eta \neq k/j, \) where \( 1 \leq k \leq \alpha - 1 \) and \( 1 \leq j \leq \alpha, \)

and, if \( \alpha \) is odd,

\( 0 < \eta \neq \alpha/j \quad \text{for} \ j = 1, 3, \ldots, \alpha. \)

Then for even integers \( 2j, 2 \leq 2j \leq \alpha, \) there are real-valued rational functions \( c_{2j} = c_{2j}(\eta) \) of \( \eta \) finite on (1.3)-(1.4), with the following property: There exists a one-to-one correspondence between solutions \( y(x) \) of (1.1) and pairs of constants \( (a_1, a_2) \) such that

\[
y = a_1 \sin \theta(x) + a_2 \cos \theta(x) + o(1),
\]

\[
y' = a_1 \cos \theta(x) - a_2 \sin \theta(x) + o(1),
\]

(1.6) \( \theta(x) = x + \sum_{2k \leq 2j \leq \alpha} c_{2j} \int_0^x h^{2j}(s) ds. \)

For related (less precise) results, see references in [3] to J. G. van der Corput.

Using a device from Hartman [2], we shall give a somewhat more

Received by the editors September 29, 1966.

1 This research was supported by the Air Force Office of Scientific Research, Office of Aerospace Research, United States Air Force, under AFOSR Contract No. F 44620-67-C-0098.
transparent proof and, at the same time, replace (1.1) by the more general equation

\[ y'' + \left( 1 + 2f(x) + 2 \sum_{m=0}^{M} h_m(x) \cos(2\eta_m x + \gamma_m) \right)y = 0. \tag{1.7} \]

It should be pointed out that Atkinson [1] had used a related method for obtaining Theorem 1.1 for \( \alpha = 2 \) and had noted that this argument can be used to show the validity of the following result for \( \alpha = 2, \ h_0 \equiv 0, \ \gamma_1 = \gamma_2 = \cdots = \gamma_M = 0. \)

**Theorem 1.2.** Let \( f(x) \in L^1[0, \infty); \ h_0(x), \cdots, h_M(x) \) functions of bounded variation on \( [0, \infty) \) for which there is an integer \( \alpha > 0 \) satisfying

\[ \sum_{m=0}^{M} \int_{0}^{\infty} |h_M'|^{\alpha+1} dx < \infty; \tag{1.8} \]

let \( \eta_0 = 0 < \eta_1 \leq \cdots \leq \eta_m \) be constants with the property that

\[ |\eta_{m(1)} \pm \eta_{m(2)} \pm \cdots \pm \eta_{m(\nu)}| \neq \tau, \quad \text{where} \ \tau = 1, \cdots, \nu, \tag{1.9} \]

whenever

\[ 0 \leq m(j) \leq M, \quad 1 \leq \nu \leq \alpha, \quad \int_{0}^{\infty} \prod_{j=1}^{\nu} |h_{m(j)}| dx = \infty; \]

finally, \( \gamma_0 = 0 \) and \( \gamma_1, \cdots, \gamma_M \) are arbitrary constants. Then there exists a one-to-one correspondence between solutions \( y(x) \) of (1.7) and pairs of constants \( (a_1, a_2) \) such that

\[ y = a_1 \sin \theta(x) + a_2 \cos \theta(x) + o(1), \]
\[ y' = a_1 \cos \theta(x) - a_2 \sin \theta(x) + o(1), \]
\[ \theta(x) = x + \int_{0}^{x} h_0 ds \]
\[ + \sum_{\mu=2}^{\alpha} \sum_{I[\mu]} c_{I[\mu]} (\cos \Gamma_{I[\mu]}) \int_{0}^{\infty} \prod_{j=1}^{\mu} h_{m(j)} ds, \tag{1.11} \]

\[ I[\mu] = (m(1), \pm m(2), \cdots, \pm m(\mu)), \]
\[ \Gamma_{I[\mu]} = \gamma_{m(1)} \pm \gamma_{m(2)} \pm \cdots \pm \gamma_{m(\mu)}, \]

\( c_{I[\mu]} = c_{m(1), \pm m(2), \cdots, \pm m(\mu)} \) are rational functions of \( (\eta_{m(1)}, \cdots, \eta_{m(\mu)}) \) which are finite for (1.9), and \( \sum_{I[\mu]} \) is the sum over the set of indices \( I[\mu] = (m(1), \pm m(2), \cdots, \pm m(\mu)) \) for which \( 0 \leq m(j) \leq M, \)

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
\[ \eta_{m(1)} \pm \eta_{m(2)} \pm \cdots \pm \eta_{m(\mu)} = 0 \quad \text{and} \quad \int_0^\infty \prod_{j=1}^\mu h_{m(j)}(s) \text{ is not convergent.} \]

Remark 1. The rational functions \( c_{m(1), \pm m(2), \ldots, \pm m(\mu)} \) are independent of the solution \( y(x) \), of the function \( f(x) \), and of the functions \( (h_0, \ldots, h_M) \) within the class of sets of functions \( (h_0, \ldots, h_M) \) for which the convergence properties of the integrals \( \int_0^\infty h_{m(j)}(x) dx \) do not vary. Note that if \( \mu = 1 \), then (1.12) can hold only for \( m(1) = 0 \).

Remark 2. If, for some \( k \) on \( 0 \leq k \leq M \), \( h_k(x) \equiv 0 \) or more generally, \( \int_0^\infty |h_k(x)| dx < \infty \), then the corresponding term \( 2h_k(x) \cos(2\eta_k x + \gamma_k) \) in (1.7) can be considered part of the term \( 2f(x) \). In this case, no \( m(j) = k \) occurs in (1.9), (1.11), and (1.12).

Remark 3. In the special case (1.1) of (1.7), we have \( h_0(x) = 0 \), \( \eta_{m(j)} = \eta \) for all \( j \geq 1 \), and (1.9) is equivalent to (1.3)–(1.4). Also the first part of (1.12) cannot hold unless \( \mu = 2j \) is even (and there are \( j \) signs + and \( j \) signs −), so that (1.10)–(1.11) reduce to (1.5)–(1.6).

2. Proof of Theorem 1.2. Introduce the abbreviation

\[ F(x) = \sum_{m=0}^M h_m(x) \cos(2\eta_m x + \gamma_m). \]

From the Prüfer transformation

\[ y(x) = r(x) \sin \phi(x), \quad y'(x) = r(x) \cos \phi(x), \]

and (1.7), we get

\[ d \log r = -F(x) \sin 2\phi dx - f(x) \sin 2\phi dx, \]

\[ d\phi = dx + F(x)(1 - \cos 2\phi) dx + f(x)(1 - \cos 2\phi) dx. \]

Following a device of Hartman [2], the last relation will also be used in the form

\[ ds = d\phi(s) - f(s)(1 - \cos 2\phi(s)) ds - F(s)(1 - \cos 2\phi(s)) ds. \]

In view of (2.2),
\begin{equation}
536 \text{PHILIP HARTMAN} [June}
\end{equation}

(2.5) \quad \log r(x) = c + o(1) + \sum_{m=0}^{M} \int_{0}^{x} h_m \cos (2\eta_m s + \gamma_m) \sin 2\phi(s) \, ds;

also, we have

\begin{equation*}
\phi(x) = x + c + o(1) + \int_{0}^{x} Fds - \int_{0}^{x} F \cos 2\phi ds.
\end{equation*}

Since $h_m$ is of bounded variation on $[0, \infty)$,

(2.6) \quad \int_{0}^{\infty} h_m(x) \cos (2\eta_m x + \gamma_m) \, dx = \lim_{T \to \infty} \int_{0}^{T} \text{ exists if } \eta_m \neq 0.

Thus

\begin{equation}
\phi(x) = x + c + o(1) + \int_{0}^{x} h_0 ds
\end{equation}

(2.7)

\begin{equation*}
- \sum_{m=0}^{M} \int_{0}^{x} h_m \cos (2\eta_m s + \gamma_m) \cos 2\phi ds.
\end{equation*}

In (2.5), (2.7) and below, $c$ will always denote a constant not necessarily the same one. The analogue of (2.6) will be used repeatedly below.

\textbf{Lemma 2.1.} Let $\phi(x)$ be as above; $g(x)$ a function of bounded variation on $[0, \infty)$, $g(x) = o(1)$ as $x \to \infty$; $\sigma, \tau, \gamma^0, \gamma$ and $\delta$ real constants such that

(2.8) \quad |\sigma| \neq |\tau|, \quad \tau \neq 0.

Then, as $x \to \infty$,

(2.9) \quad \int_{0}^{x} g(s) \cos(2\sigma s + \gamma^0 - \gamma) \cos(2\tau \phi - \delta) \, ds

\begin{equation*}
= c + o(1) + 4(\tau^2 - \sigma^2)^{-1}\{ \cdots \},
\end{equation*}

where $\{ \cdots \}$ is the expression

\begin{equation}
\{ \cdots \} = (\sigma \tau) \sum_{m=0}^{M} \sum_{j=0}^{1} \sum_{k=-1}^{1} \epsilon_k \int_{0}^{x} g h_m
\end{equation}

\begin{equation*}
\times \sin [2(\sigma + (-1)^j\eta_m) s + \gamma^0 - \gamma + (-1)^j\gamma_m] \times \cos [2(\sigma + (-1)^j\eta_m) s + \gamma^0 - \gamma + (-1)^j\gamma_m]
\end{equation*}

\begin{equation*}
\times \cos [2(\tau + k) \phi - \delta] ds + \tau^2 \sum_{m=0}^{M} \sum_{j=0}^{1} \sum_{k=-1}^{1} \epsilon_k \int_{0}^{x} g h_m.
\end{equation*}
\( \epsilon_{\pm 1} = 1 \) and \( \epsilon_0 = -2 \).

We shall only need the cases \( \gamma = \delta = 0 \) and \( \gamma = \delta = \pi/2 \) for the asymptotic behavior of \( \phi(x) \), and the cases \( \gamma = \pi/2, \delta = 0 \) and \( \gamma = 0, \delta = \pi/2 \) for \( r(x) \).

**Proof.** Let \( I \) denote the integral on the left of (2.9). Replace \( ds \) in \( I \) by its value in (2.4) and integrate the resulting first term by parts to obtain

\[
I = c + o(1) + (\sigma/\tau) \int_0^x g(s) \sin(2\sigma s + \gamma^0 - \gamma) \sin(2\tau \phi - \delta) ds
\]

(2.11)

\[
- \int_0^x g(s) \cos(2\sigma s + \gamma^0 - \gamma) \cos(2\tau \phi - \delta) F(s)(1 - \cos 2\phi) ds.
\]

Use (2.4) in the first integral on the right of (2.11) and integrate the first term by parts,

\[
I = c + o(1) + (\sigma/\tau)^2 I
\]

\[
- (\sigma/\tau) \int_0^x g(s) \sin(2\sigma s + \gamma^0 - \gamma) \sin(2\tau \phi - \delta) F(s)(1 - \cos 2\phi) ds
\]

\[
- \int_0^x g(s) \cos(2\sigma s + \gamma^0 - \gamma) \cos(2\tau \phi - \delta) F(s)(1 - \cos 2\phi) ds.
\]

In view of the relations, for \( \chi = \sin \) or \( \chi = \cos \),

\[
2\chi(2\tau \phi - \delta)(1 - \cos 2\phi)
\]

\[
= - \sum_{k=-1}^{1} \epsilon_k \chi[2(\tau + k) \phi - \delta],
\]

and

\[
2\chi(2\sigma s + \gamma^0 - \gamma) F(s)
\]

\[
= \sum_{m=0}^{M} \sum_{j=0}^{1} h_m(s) \chi[2(\sigma + (-1)^j \eta_m) s + \gamma^0 - \gamma + (-1)^j \gamma_m],
\]

formula (2.9) follows. This completes the proof.

**On \( \phi(x) \).** We now show, by an induction on \( \nu \) for \( 1 \leq \nu \leq \alpha + 1 \), that, under the assumption (1.9), \( \phi(x) \) has the form
\[\phi(x) = x + c + o(1) + \int_0^x h_0 ds + \sum_{\mu=1}^{r-1} \sum_{I[\mu]} c_{I[\mu]} \left( \cos \Gamma_{I[\mu]} \int_0^x \prod_{j=1}^{\mu} h_{m(j)} ds \right) \]

\[(2.12) + \sum_{\tau=1}^\nu \sum_{I[\mu]}' a_{\tau, I[\nu]} \int_0^x \prod_{j=1}^\nu h_{m(j)} \times \cos(2N_{I[\nu]} s + \Gamma_{I[\nu]} \cos 2\tau \phi ds) + \sum_{\tau=1}^\nu \sum_{I[\nu]}' b_{\tau, I[\nu]} \int_0^x \prod_{j=1}^\nu h_{m(j)} \times \sin(2N_{I[\nu]} s + \Gamma_{I[\nu]} \sin 2\tau \phi ds),\]

where

\[N_{I[\nu]} = \eta_{m(1)} \pm \eta_{m(2)} \pm \cdots \pm \eta_{m(\nu)},\]

\[c_{I[\mu]} = c_{m(1), \pm m(2), \cdots, \pm m(\mu)} \quad \text{and} \quad a_{\tau, I[\nu]} = a_{\tau, m(1), \pm m(2), \cdots, \pm m(\nu)}, \quad b_{\tau, I[\nu]} = b_{\tau, m(1), \pm m(2), \cdots, \pm m(\nu)} \]

are rational functions of \((\eta_{m(1)}, \cdots, \eta_{m(\mu)})\) and \((\eta_{m(1)}), \cdots, \eta_{m(\nu)})\), respectively, finite for (1.9); \(\sum_{I[\mu]}\) is the sum over the sets of indices \((m(1), \pm m(2), \cdots, \pm m(\mu)), 0 \leq m(j) \leq M,\)

for which

\[(2.13) \eta_{m(1)} \pm \cdots \pm \eta_{m(\mu)} = 0 \quad \text{and} \quad \int_0^\infty \prod_{j=1}^{\mu} h_{m(j)} dx \text{ is not convergent;}\]

finally \(\sum_{I[\nu]}'\) is the sum over all sets \(I[\nu] = (m(1), \pm m(2), \cdots, \pm m(\nu))\) for which

\[(2.14) \quad \int_0^\infty \prod_{j=1}^\nu | h_{m(j)} | dx = \infty.\]

The formula (2.7) can be written in the form (2.12) for \(\nu = 1\). We assume (2.12) for some given \(\nu, 1 \leq \nu \leq \alpha\). Then the assumption (1.9) makes Lemma 2.1 applicable to each term in the last two sums of (2.12), with \(\gamma = \delta = \pi/2\) in the last sum and \(\gamma = \delta = 0\) in the next to last sum, \(\sigma = \eta_{m(1)} \pm \cdots \pm \eta_{m(\nu)}, \quad \gamma^0 = \gamma_{m(1)} \pm \cdots \pm \gamma_{m(\nu)}, \quad \text{and} \quad g(x) = \prod_{j=1}^\nu h_{m(j)} \text{ for } j = 1, \cdots, \nu. \)

This shows the validity of (2.12) when \(\nu\) is replaced by \(\nu + 1\), since

\[\int_0^x \prod_{j=1}^{\nu+1} h_{m(j)} \cos(2N_{I[\nu+1]} s + \Gamma_{I[\nu+1]}) ds = c + o(1)\]
if the first part of (2.13) does not hold. Hence (2.12) is valid for \( \nu = \alpha + 1 \). Thus \( \phi(x) \) can be written in the form

\[
\phi(x) = c^0 + o(1) + \theta(x),
\]

where \( \theta(x) \) is independent of the solution \( y(x) \) and is given by (1.11).

On \( r(x) \). Starting with (2.5), the cases \( \gamma = \pi / 2, \delta = 0 \) and \( \gamma = 0, \delta = \pi / 2 \) of Lemma 2.1 imply, by an induction on \( \nu \), \( 1 \leq \nu \leq \alpha + 1 \), that \( \log r(x) \) can be written in the form

\[
\log r(x) = c + o(1)
\]

\[
+ \sum_{\nu=1}^{\nu-1} \sum_{I[\nu]} c_{I[\nu]}^*(\sin \Gamma_{I[\nu]}) \int_0^x \prod_{j=1}^\mu h_{m(j)} ds
\]

\[
+ \sum_{\tau=1}^{\nu-1} \sum_{I[\tau]} d_{\tau,I[\nu]}^*
\]

\[
\times \int_0^x \prod_{j=1}^\nu h_{m(j)} \cos(2N_{I[\nu]} s + \Gamma_{I[\nu]}) \sin 2\tau \phi ds
\]

\[
+ \sum_{\tau=1}^{\nu-1} \sum_{I[\tau]} b_{\tau,I[\nu]}^*
\]

\[
\times \int_0^x \prod_{j=1}^\nu h_{m(j)} \sin(2N_{I[\nu]} s + \Gamma_{I[\nu]}) \cos 2\tau \phi ds,
\]

in notation analogous to (2.12).

Thus, the case \( \nu = \alpha + 1 \) shows that

\[
r(x) = [c^1 + o(1)] \exp \rho(x),
\]

where \( \rho(x) \) is independent of the solution \( y(x) \) and

\[
\rho(x) = \sum_{\mu=1}^\alpha \sum_{I[\mu]} c_{I[\mu]}^*(\sin \Gamma_{I[\mu]}) \int_0^x \prod_{j=1}^\mu h_{m(j)} ds.
\]

Completion of the proof. Let \( y_1(x), y_2(x) \) be two solutions of (1.7) with the Wronskian

\[
y_1 y_2' - y_1' y_2 = 1.
\]

Then, by (2.1), (2.15) and (2.17), for \( j = 1, 2 \),

\[
y_j = [c_j^1 + o(1)] e^{\rho(x)} \sin [c_j^0 + o(1) + \theta(x)],
\]

\[
y_j' = [c_j^1 + o(1)] e^{\rho(x)} \cos [c_j^0 + o(1) + \theta(x)],
\]

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
where \((c_j^0, c_j^1)\) are the constants \((c^0, c^1)\) belonging to \(y_j(x)\). By (2.19),

\[
\left[ c_1^1 c_2^1 + o(1) \right] e^{2p(x)} \sin (c_1^0 - c_2^0 + o(1)) = 1.
\]

Thus, according as

\[
(2.21) \quad c_1^1 c_2^1 \sin (c_1^0 - c_2^0) = 0 \quad \text{or} \quad \neq 0,
\]

it follows that

\[
(2.22) \quad \lim_{x \to \infty} \rho(x) = +\infty \quad \text{or} \quad \text{exists (finite)}.
\]

Actually, the first alternative in (2.22) cannot hold. In order to see this, consider the differential equation obtained by changing the signs of the \(\gamma_m\) in (1.7),

\[
y'' + \left[ 1 + 2f(x) + 2 \sum_{m=0}^M h_m(x) \cos (2\gamma_m x - \gamma_m) \right] y = 0,
\]

and let \(\theta_1(x), \rho_1(x)\) belong to this equation as \(\theta(x), \rho(x)\) in (1.11), (2.18) belong to (1.7). Then, the deduction of (2.22) shows that

\[
\lim_{x \to \infty} \rho_1(x) = +\infty \quad \text{or} \quad \text{exists (finite)}.
\]

But \(\theta(x) = \theta_1(x)\) and \(\rho_1(x) = -\rho(x)\); thus \(\rho(\infty) \neq +\infty\). Consequently, changing \(c^1\), (2.17) becomes \(r(x) = c^1 + o(1), \rho(x) = 0\).

Correspondingly, by the formulae following (2.19),

\[
y_j = \left[ c_j^1 + o(1) \right] \sin (c_j^0 + o(1) + \theta(x)),
\]

\[
y_j' = \left[ c_j^1 + o(1) \right] \cos (c_j^0 + o(1) + \theta(x))
\]

and \(c_1^1 c_2^1 \sin (c_1^0 - c_2^0) \neq 0\). Thus, for a solution \(y(x) \neq 0\), \(c^1 > 0\) in \(r(x) = c^1 + o(1)\), and linearly independent solutions \(y_1, y_2\) belong to pairs \((c_1^0, c_1^1), (c_2^0, c_2^1)\) with \(c_1^0 \neq c_2^0\) (mod \(\pi\)). This completes the proof of Theorem 1.2.

References


The Johns Hopkins University