THE DISTRIBUTION OF \( k \)TH POWER RESIDUES
AND NONRESIDUES

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1. Introduction. In 1962 D. A. Burgess [1] established this general theorem concerning character sums:

**Theorem A.** If \( p \) is a prime and if \( \chi \) is a nonprincipal Dirichlet character, modulo \( p \), and if \( H \) and \( r \) are arbitrary positive integers then

\[
\sum_{m=n+1}^{n+H} \chi(m) \ll H^{1-1/(r+1)}p^{1/4r} \ln p
\]

for any integer \( n \), where \( A \ll B \) is Vinogradov's notation for \( |A| \leq cB \) for some constant \( c \), and in this theorem \( c \) is absolute.

In this paper Theorem A will be used to improve a special case of the Vinogradov result [3]:

**Theorem B.** Let \( \mathcal{E}_j \) be a class of \( k \)th power nonresidues for \( j = 1, 2, 3, \ldots, k \) and let \( \mathcal{E}_0 \) be the class of \( k \)th power residues, \( k = p - 1 \). Also let \( N_j(H) \) be the number of positive integers in \( \mathcal{E}_j \) that are \( \leq H \). Then \( N_j(H) = H/k + T_j \) where \( T_j < T + p/2 \) with

\[
T = \sum_{x=1}^{H} \sum_{(x,y)=1}^{p/x} (p/xy + 1).
\]

In particular, Theorem B implies that \( T_j < \sqrt{p} \ln p \).

Specifically, in this paper the following is proved:

**Theorem.** Let \( \mathcal{E}_j \) be the classes of \( k \)th power nonresidues, \( j = 1, 2, 3, \ldots, k - 1 \); and \( \mathcal{E}_0 \) is the class of \( k \)th power residues, \( k = p - 1 \). Also let \( N_j(H) \) be the number of positive integers in \( \mathcal{E}_j \) that are \( \leq H \). Then \( N_j(H) = H/k + T_j \) where \( T_j \ll H^{1-1/(r+1)}p^{1/4r} \ln p \), \( r \) is a positive integer.

Notice that this theorem is significant for \( p^{1/4+1/4r}(\ln p)^{r+1}\ll H \) and is an improvement of Theorem B for \( H < p^{1/2+1/4r-1/4r^2} \). Theorem B has content only when \( \sqrt{p} \ln p < H \).

In [2] the author proved this theorem for \( k = 3 \) and 5 but erroneously referred to these as being special cases of Theorem B.

Received by the editors February 21, 1967.

*Research for this paper was supported in part by NSF Grant No. GP-6227 and WSU Grant-in-aid for research; project number 728.*

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2. **Proof of the Theorem.** Let $\chi$ be a $k$th power Dirichlet character and order the $E_j$ so that $\chi(a) = \rho^j$ for all $a$ in $E_j$, where $\rho$ is a primitive $k$th root of unity. Let

$$N_j(H) = H/k + T_j.$$  

Now since $\sum_{j=0}^{k-1} N_j(H) = \sum_{j=0}^{k-1} (H/k + T_j) = H + \sum_{j=0}^{k-1} T_j$ and trivially $\sum_{j=0}^{k-1} N_j(H) = H$, we have therefore

$$\sum_{j=0}^{k-1} T_j = 0.$$  

Also

$$\sum_{m=1}^{H} \chi(m) = \sum_{j=0}^{k-1} \rho^j N_j(H) = \sum_{j=0}^{k-1} \rho^j H/k + \sum_{j=0}^{k-1} \rho^j T_j = H/k \sum_{j=0}^{k-1} \rho^j + \sum_{j=0}^{k-1} \rho^j T_j = \sum_{j=0}^{k-1} \rho^j T_j.$$  

And by Burgess' Theorem this implies

$$\left| \sum_{j=0}^{k-1} \rho^j T_j \right| < cH^{1-1/(r+1)} \rho^{1/4r} \ln \rho.$$  

Now $\chi^t$ is also a nonprincipal Dirichlet character for $1 \leq t \leq k-1$, and since $\chi^t(a) = \rho^{tj}$ for $a$ in $E_j$ it follows that:

$$\sum_{m=1}^{H} \chi^t(m) = \sum_{j=0}^{k-1} \rho^{tj} N_j(H) = \sum_{j=0}^{k-1} \rho^{tj} H/k + \sum_{j=0}^{k-1} \rho^{tj} T_j = H/k \sum_{j=0}^{k-1} \rho^{tj} + \sum_{j=0}^{k-1} \rho^{tj} T_j = \sum_{j=0}^{k-1} \rho^{tj} T_j.$$  

Applying Burgess' Theorem one has:

$$\left| \sum_{j=0}^{k-1} \rho^{tj} T_j \right| < c_j H^{1-1/(r+1)} \rho^{1/4r} \ln \rho, \quad 1 \leq t < k.$$  

Now for a specific $E_j$, say $E_j*$, consider expression (IV) divided by $\rho^{j*}$ yielding:

$$\left| \sum_{j=0}^{k-1} \rho^{(j-j*)} T_j \right| < c_i H^{1-1/(r+1)} \rho^{1/4r} \ln \rho, \quad 1 \leq t < k.$$
Now summing over all expressions in (V) and throwing in expression (III) one has:

\[(VI) \left| \sum_{t=0}^{k-1} \sum_{j=0}^{k-1} \rho_t(j-j^*) T_j \right| \leq \sum_{t=0}^{k-1} \left| \sum_{j=0}^{k-1} \rho_t(j-j^*) T_j \right| .\]

But

\[
\sum_{t=0}^{k-1} \sum_{j=0}^{k-1} \rho_t(j-j^*) T_j = \sum_{j=0}^{k-1} T_j \sum_{t=0}^{k-1} \rho_t(j-j^*)
\]

\[
= \sum_{j=0; j \neq j^*}^{k-1} T_j \sum_{t=0}^{k-1} \rho_t(j-j^*) + \sum_{t=0}^{k-1} T_j^*
\]

\[= kT_j^*, \text{ since } \sum_{t=0}^{k-1} \rho_t(j-j^*) = 0 \text{ unless } j = j^*.\]

Hence

\[|T_j^*| < \frac{1}{k} \sum_{t=1}^{k-1} c_t H^{1-1/(r+1)} p^{1/4r} \ln p
\]

\[= c^* H^{1-1/(r+1)} p^{1/4r} \ln p.\]

**BIBLIOGRAPHY**


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