A CRITERION FOR n-n OSCILLATIONS IN Differential Equations of Order 2n

DON B. HINTON

For the second-order equation \( y'' + qy = 0 \), Wintner [6] proved that a sufficient condition for oscillation was that

\[
(1) \quad t^{-1} \int_{0}^{t} q(x)(t - x)dx \to \infty \quad \text{as} \quad t \to \infty.
\]

Independently Leighton [2] proved that a sufficient condition for oscillation of \( (ry')' + qy = 0 \) was that \( q \) be positive for sufficiently large \( x \) and that

\[
(2) \quad \int_{0}^{\infty} r(x)^{-1}dx = \infty \quad \text{and} \quad \int_{0}^{\infty} q(x)dx = \infty.
\]

Subsequently Leighton [3] proved that conditions (2) were sufficient without the restriction \( q \) be positive for sufficiently large \( x \).

In this paper we prove analogous theorems for the linear equation of order \( 2n \) \( (ry^{(n)})^{(n)} + (-1)^{n-1}qy = 0 \). There will be no sign restrictions on the function \( q \). For \( n = 1 \), the earlier results of Wintner and Leighton will be contained in our theorems. The results of this paper parallel earlier results of the author for a fourth-order equation with middle term [1].

Throughout \( r \) and \( q \) denote continuous, real-valued functions on a ray \( [a, \infty) \) and \( r \) is assumed to be positive-valued. If the real-valued function \( y \) has \( n \) continuous derivatives and \( ry^{(n)} \) has \( n \) continuous derivatives, then we define \( L(y) \) by:

\[
(3) \quad L(y) = (ry^{(n)})^{(n)} + (-1)^{n-1}qy.
\]

Such a \( y \) is said to be admissible for \( L \).

The operator \( L \) is called oscillatory on \( [a, b] \) if and only if there is an admissible function \( y \), \( y \neq 0 \), and numbers \( c \) and \( d \), \( a \leq c < d \leq b \), such that \( L(y) = 0 \) and

\[
(4) \quad y(c) = \cdots = y^{(n-1)}(c) = 0 = y(d) = \cdots = y^{(n-1)}(d).
\]

Otherwise \( L \) is called nonoscillatory on \( [a, b] \).

For \( b > a \), let \( \omega(b) \) denote the set of all real-valued \( y \) on \( [a, b] \) such

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511
that (a) $y$ has $n-1$ continuous derivatives on $[a, b]$ with $y^{(n-1)}$ absolutely continuous, (b) $y^{(n)}$ is essentially bounded ($y^{(n)}$ denotes the almost everywhere derivative of $y^{(n-1)}$) and (c) $y$ satisfies the boundary conditions (4) with $a = c$ and $b = d$.

The function $I_b$ is defined on $\mathcal{C}(b)$ by:

$$I_b(y) = \int_a^b \left[ r(x) y^{(n)}(x)^2 - q(x) y(x)^2 \right] dx.$$  \hfill (5)

For our basic criterion of oscillation we consider a vector-matrix formulation of $L(y) = 0$. Let the $n \times n$ matrices of functions $A = [a_{ij}]$, $B = [b_{ij}]$ and $C = [c_{ij}]$ be defined by:

$$a_{ij} = 0 \quad \text{if } j - i \neq 1, \quad b_{ij} = 0 \quad \text{if } i \neq n \text{ or } j \neq n$$

$$= 1 \quad \text{if } j - i = 1, \quad = 1/r \quad \text{if } i = n \text{ and } j = n$$

and

$$c_{ij} = 0 \quad \text{if } i \neq 1 \text{ or } j \neq 1,$$

$$= -q \quad \text{if } i = 1 \text{ and } j = 1.$$  \hfill (6)

Then if $L(y) = 0$,

$$\eta = \left[ y^{(i-1)} \right]_{i=1}^n \quad \text{and} \quad \xi = \left[ (-1)^{n-i} (r y^{(n)})^{(n-i)} \right]_{i=1}^n,$$

it is readily verified that

$$\eta' = A \eta + B \xi, \quad \xi' = C \eta - A^T \xi$$  \hfill (7)

where $A^T$ denotes the transpose of $A$.

Conversely, if $(\eta, \xi)$ is a pair of absolutely continuous real vector-valued functions on $[a, b]$ such that (7) hold almost everywhere, then it follows that (7) hold everywhere, and the first component $\eta_1$ of $\eta$ is admissible for $L$ and $L(\eta_1) = 0$.

Reid [5, p. 673] has defined the system (7) to be oscillatory on $[a, b]$ if and only if there is a pair $(\eta, \xi)$ of absolutely continuous real or complex vector-valued functions on $[a, b]$ such that (7) hold almost everywhere, $\eta \neq 0$, and there are numbers $c$ and $d$, $a \leq c < d \leq b$, such that $\eta(c) = 0 = \eta(d)$. The one-to-one correspondence between solutions $y$ of $L(y) = 0$ and the first components $\eta_1$ of solutions of (7) proves that $L$ is oscillatory on $[a, b]$ if and only if (7) is oscillatory on $[a, b]$.

We remark that in the terminology of Reid [5, p. 673], the system (7) is identically normal on every subinterval of $[a, \infty)$ since if
(η, ξ) is a solution and η = 0, then the first equation of (7) implies
ξ_n = 0, and the second equation of (7) implies successively that
ξ_{n-1} = 0, ..., ξ_1 = 0.

Our basic criterion for oscillation is the following:

**Theorem 1.** If there exists a y ∈ α(b), y ≠ 0, such that I_b(y) ≤ 0, then
L is oscillatory on [a, b].

**Proof.** We note first that C(x)^T = C(x), B(x)^T = B(x) and that
B(x) is nonnegative definite on [a, b]. Moreover, if y ∈ α(b), η
= \{y^{(i-1)}\}_{i=1}^n, ξ = \{ξ_i\} where ξ_n = ry^{(n)} and ξ_i = 0 otherwise, then

\[ ξ^T B_ξ + η^T C_η = r(y^{(n)})^2 - qy^2. \]

Then in the terminology of Reid [5, p. 678], we have (η, ξ) ∈ D_0 [a, b],
η ≠ 0 and I[η, ξ; a, b] = I_b(y) ≤ 0. Thus by Theorem 5.2 of [5], L is
oscillatory on [a, b].

From Theorem 1 we obtain a comparison theorem for oscillation.

**Theorem 2.** If r_1 and q_1 are continuous, real-valued functions on
[a, b] with r_1 positive-valued, y ∈ α(b) is a nontrivial solution of L(y) = 0
and L_1(y) = (r_1y^{(n)}) + (-1)^{n-1}q_1y, then L_1 is oscillatory on [a, b] if

\[ \int_a^b [(r(x) - r_1(x))y^{(n)}(x)^2 - (q(x) - q_1(x))y(x)^2]dx ≥ 0. \]

**Proof.** Let J_b be defined by the right-hand side of (5) where r and
q are replaced by r_1 and q_1 respectively. Then equation (8) reduces to
I_b(y) - J_b(y) ≥ 0. Integrating \( \int_a^b (r_1y^{(n)}(x)) \cdot y^{(n)}(x)dx \) by parts n times
proves that I_b(y) = (-1)^n \( \int_a^b L(y) \cdot y(x)dx \). Since L(y) = 0, equation
(8) is equivalent to J_b(y) ≥ 0. Theorem 1 now implies L_1 is oscillatory
on [a, b].

As a corollary we have a generalization of the Sturm-Picone The-
orem for second-order equations.

**Corollary 2.1.** If L is oscillatory on [a, b] and r_1(x) ≤ r(x) and
q_1(x) ≥ q(x) for each x in [a, b], then L_1 is oscillatory on [a, b].

We now prove our principal theorem.

**Theorem 3.** If there is a positive-valued continuous function h on
[a, ∞) such that as t → ∞,
(i) \( \int_a^t x^{n-1}h(x)dx \) → ∞ and
(ii) \( \lim \inf J(t) = -∞ \)

where
\[ J(t) = \left[ \int_a^t \left\{ \frac{d}{dx} \left[ (n-1) \frac{1}{2} h(x) \right]^2 - q(x) \left( \int_x^t (s-x)^{n-1} h(s) \, ds \right)^2 \right\} \, dx \right] \cdot \left[ \int_a^t x^{n-1} h(x) \, dx \right]^{-2}, \]

then there is a number \( b > a \) such that \( L \) is oscillatory on \([a, b]\).

Proof. For each number \( t > a + 1 \) we construct a function \( y_t \) on \([a, t]\) such that \( y_t \in \mathcal{A}(t) \). For some \( t \) sufficiently large, we will have \( I_t(y_t) < 0 \), thus proving Theorem 3.

For \( t > a + 1 \), define \( z_t \) on \([a, t]\) by

\[ z_t(x) = \left[ \int_x^t (s-x)^{n-1} h(s) \, ds \right] \left[ \int_a^t s^{n-1} h(s) \, ds \right]^{-1}. \]

It is clear that for \( k = 0, \ldots, n-1, \]

\[ \left[ \frac{d^k z_t(x)}{dx^k} \right]_{x=t} = 0. \]

For \( k = 0, \ldots, n-1 \), let

\[ c_{tk} = \left[ \frac{d^k z_t(x)}{dx^k} \right]_{x=a+1}. \]

Application of l'Hospital's rule proves that \( c_{t0} \rightarrow 1 \) and for \( k = 1, \ldots, n-1, \)

\[ c_{tk} \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty. \]

Let \( p_t \) be the polynomial

\[ p_t(x) = (x-a)^n \sum_{j=0}^{n-1} a_j x^j \]

satisfying, for \( k = 0, \ldots, n-1, \)

\( d^k \left[ \frac{dp_t(x)}{dx^k} \right]_{x=a+1} = c_{tk}. \)

A simple calculation proves that for \( k = 0, \ldots, n-1, \)

\[ \left[ \frac{d^k p_t(x)}{dx^k} \right]_{x=a} = 0. \]

The \( n \) coefficients \( a_0, \ldots, a_{n-1} \) are determined as solutions to the \( n \)
linear equations (9). The matrix of coefficients does not depend on \( t \).
That the determinant of the matrix of coefficients is nonzero follows
from Theorem II of [4] and the fact that \( p_t \) is a solution of the dif-
ferential equation \( y^{(2n)} = 0. \) Hence \( a_0, \ldots, a_{n-1} \) are bounded func-
tions of \( t \). Define \( y_t \) by:

\[ y_t(x) = p_t(x) \quad \text{for} \quad a \leq x \leq a + 1 \]

and \( y_t(x) = z_t(x) \quad \text{for} \quad a + 1 < x \leq t. \)

Then \( y_t \in \mathcal{A}(t) \).
We have $I_t(y_t) = P(t) + K(t)$, where

$$P(t) = \int_a^{a+1} \left[ r(x)\left(\frac{d^n}{dx^n} p_t(x)\right)^2 - q(x)p_t(x)^2 \right] dx$$

and

$$K(t) = \int_{a+1}^t \left[ r(x)(z_t^{(n)}(x))^2 - q(x)z_t(x)^2 \right] dx.$$

Since $a_0, \ldots, a_{n-1}$ are bounded functions of $t$, $P(t) = O(1)$ as $t \to \infty$. If $M$ is a bound for $r$, $q$ and $h$ on $[a, a+1]$, then

$$\left| \int_a^{a+1} \left\{ r(x)[(n-1)!h(x)]^2 - q(x) \left[ \int_x^t (s-x)^{n-1}h(s)ds \right]^2 \right\} dx \right| \leq M \left\{ [(n-1)!M]^2 + \left[ \int_a^t (s-a)^{n-1}h(s)ds \right]^2 \right\}. $$

Hence as $t \to \infty$,

$$\int_a^{a+1} \left[ r(z_t^{(n)}(x))^2 - q(x)z_t(x)^2 \right] dx = O(1).$$

Thus condition (ii) implies that $\lim \inf K(t) = -\infty$ as $t \to \infty$. Hence $\lim \inf I_t(y_t) = -\infty$ as $t \to \infty$. In particular, $I_t(y_t) < 0$ for some sufficiently large $t$, thus proving Theorem 3.

For $h=1$, we have a useful corollary of Theorem 3.

**Corollary 3.1.** If

$$\lim sup \frac{1}{t} \int_a^t r(x)dx < \infty,$$

and

$$\lim \frac{1}{t} \int_a^t q(x)(t-x)^{2n}dx = \infty,$$

then for some $b > a$, $L$ is oscillatory on $[a, b]$.

A weaker but more applicable version of Theorem 3 may be stated as follows:

**Theorem 4.** If there is a positive-valued continuous function $h$ on $[a, \infty)$ such that as $t \to \infty$

(i) $\int_a^t x^{n-1}h(x)dx \to \infty$,

(ii) $\lim sup \left\{ \int_a^t r(x)h(x)^2dx \right\} \left\{ \int_a^t x^{n-1}h(x)dx \right\}^{-2} < \infty$ and
(iii) \( t^{1-n} \int_{a}^{t} q(x)(t-x)^{n-1}dx \to \infty, \)

then there is a number \( b > a \) such that \( L \) is oscillatory on \([a, b]\).

**Proof.** Our proof will consist of proving that (i), (ii) and (iii) of Theorem 4 imply (ii) of Theorem 3. First we prove two lemmas.

**Lemma 1.** If \( f \) is a continuous, real-valued function on \([a, \infty)\) such that for some integer \( p \geq 0, \)

\[
\lim_{t \to \infty} t^{-p} \int_{a}^{t} f(x)(t-x)^{p}dx = \infty,
\]

then for each integer \( k > p, \)

\[
\lim_{t \to \infty} t^{-k} \int_{a}^{t} f(x)(t-x)^{k}dx = \infty.
\]

**Proof.** A straightforward inductive proof using l'Hospital's rule is omitted.

**Lemma 2.** Suppose that (i) and (iii) of Theorem 4 hold, and that for \( i = 1, \ldots, n, \)

\[Q_i(t) = \int_{a}^{t} q(x) \left( \int_{a}^{t} (s-x)^{n-1}h(s)ds \right) (t-x)^{i-1}dx.\]

Then for \( i = 1, \ldots, n \) and as \( t \to \infty, \)

\[
Q^i(t)/t^{i-1} \int_{a}^{t} x^{n-1}h(x)dx \to \infty.
\]

**Proof.** For \( i = 1 \) we have by (iii),

\[
\lim_{t \to \infty} Q_i'(t)/t^{n-1}h(t) = \lim_{t \to \infty} t^{1-n} \int_{a}^{t} q(x)(t-x)^{n-1}dx = \infty.
\]

Hence (10) holds.

Suppose (10) holds for some \( i, 1 \leq i < n. \) Let \( M \) be a positive number. Since

\[Q_{i+1}(t) = iQ_i(t) + \int_{a}^{t} q(x)(t-x)^{n+i-1}h(t)dx,\]

an application of the inductive hypothesis and Lemma 1 yields that for sufficiently large \( t, \) say \( t \geq t_0, \)

\[Q_{i+1}'(t) \geq Mt^{i-1} \int_{a}^{t} x^{n-1}h(x)dx + Mt^{n+i-1}h(t).\]
Hence for \( t > t_0 \),
\[
Q_{i+1}(t) \geq Q_{i+1}(t_0) + M \left\{ \int_{t_0}^{t} \left[ is^{i-1} \int_{a}^{s} x^{n-1} h(x) \,dx + s^{n+i-1} h(s) \right] \,ds \right\}
\]
\[
= Q_{i+1}(t_0) + M \left( \int_{a}^{t} x^{n-1} h(x) \,dx \right) (t - t_0) + Mt \int_{t_0}^{t} x^{n-1} h(x) \,dx.
\]
The above inequality implies
\[
\lim_{t \to -\infty} \inf \frac{Q_{i+1}(t)}{t} \int_{a}^{t} x^{n-1} h(x) \,dx \geq M,
\]
from which we conclude that (10) holds for \( i+1 \).

That (ii) of Theorem 3 is a consequence of (i), (ii) and (iii) of Theorem 4 now follows by applying l'Hospital's rule and Lemma 2 to
\[
\lim_{t \to -\infty} \left\{ \int_{a}^{t} q(x) \left\{ \int_{a}^{t} (s - x)^{n-1} h(s) \,ds \right\}^{2} \,dx \right\} \left\{ \int_{a}^{t} x^{n-1} h(x) \,dx \right\}^{-2}
\]
\[
= \lim_{t \to -\infty} Q_{i}(t) / t^{n-1} \int_{a}^{t} x^{n-1} h(x) \,dx = \infty.
\]
For \( h = 1/r \), condition (i) implies (ii), and we obtain the following special case of Theorem 4.

**Corollary 4.1.** If \( \int_{a}^{\infty} x^{n-1} r(x)^{-1} \,dx = \infty \) and
\[
\lim_{t \to -\infty} t^{1-n} \int_{a}^{t} q(x) (t - x)^{n-1} \,dx = \infty,
\]
then there is a number \( b > a \) such that \( L \) is oscillatory on \([a, b]\).

For \( n = 1 \), Corollary 4.1 gives the sufficient criterion of Leighton [3].

We note that Lemma 1 and Corollary 3.1 imply the previously mentioned result of Wintner for the equation \( y'' + qy = 0 \). That the condition
\[
t^{-2} \int_{a}^{t} q(x) (t - x)^{2} \,dx \to \infty \text{ as } t \to \infty
\]
is more general than Wintner's condition is shown by the following example.

**Example.** Let \( w(t) = t^{2} \sin^{2} t \) and let \( q = w'' \). It then follows that
\[
\int_{0}^{t} q(x) \,dx = w'(t) = 2t \sin^{2} t + t^{2} \sin 2t,
\]
\[ t^{-1} \int_0^t q(x)(t - x)dx = t \sin^2 t \]

and

\[ t^{-2} \int_0^t q(x)(t - x)^2dx = \frac{t}{3} - \left(\frac{1}{2} - \frac{1}{4t^2}\right) \sin 2t - \frac{(\cos 2t)}{2t}. \]

Hence by Corollary 3.1, \( L(y) = y'' + qy \) is oscillatory.

We remark that Theorem 4 is not applicable in this example since condition (iii) does not hold.

References


University of Georgia