A CRITERION FOR n-n OSCILLATIONS IN DIFFERENTIAL EQUATIONS OF ORDER 2n

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For the second-order equation $y'' + qy = 0$, Wintner [6] proved that a sufficient condition for oscillation was that

$$t^{-1} \int_{t}^{\infty} q(x)(t - x)dx \to \infty \quad \text{as} \quad t \to \infty.$$  

Independently Leighton [2] proved that a sufficient condition for oscillation of $(ry')' + qy = 0$ was that $q$ be positive for sufficiently large $x$ and that

$$\int_{0}^{\infty} r(x)^{-1} dx = \infty \quad \text{and} \quad \int_{0}^{\infty} q(x) dx = \infty.$$  

Subsequently Leighton [3] proved that conditions (2) were sufficient without the restriction $q$ be positive for sufficiently large $x$.

In this paper we prove analogous theorems for the linear equation of order $2n$ $(ry^{(n)}(n) + (-1)^{n-1}qy = 0$. There will be no sign restrictions on the function $q$. For $n = 1$, the earlier results of Wintner and Leighton will be contained in our theorems. The results of this paper parallel earlier results of the author for a fourth-order equation with middle term [1].

Throughout $r$ and $q$ denote continuous, real-valued functions on a ray $[a, \infty)$ and $r$ is assumed to be positive-valued. If the real-valued function $y$ has $n$ continuous derivatives and $ry^{(n)}$ has $n$ continuous derivatives, then we define $L(y)$ by:

$$L(y) = (ry^{(n)}(n) + (-1)^{n-1}qy.$$  

Such a $y$ is said to be admissible for $L$.

The operator $L$ is called oscillatory on $[a, b]$ if and only if there is an admissible function $y$, $y \neq 0$, and numbers $c$ and $d$, $a \leq c < d \leq b$, such that $L(y) = 0$ and

$$y(c) = \cdots = y^{(n-1)}(c) = 0 = y(d) = \cdots = y^{(n-1)}(d).$$  

Otherwise $L$ is called nonoscillatory on $[a, b]$.

For $b > a$, let $\mathcal{A}(b)$ denote the set of all real-valued $y$ on $[a, b]$ such
that (a) \( y \) has \( n-1 \) continuous derivatives on \([a, b]\) with \( y^{(n-1)} \) absolutely continuous, (b) \( y^{(n)} \) is essentially bounded (\( y^{(n)} \) denotes the almost everywhere derivative of \( y^{(n-1)} \)) and (c) \( y \) satisfies the boundary conditions (4) with \( a = c \) and \( b = d \).

The function \( I_b \) is defined on \( \alpha(b) \) by:

\[
I_b(y) = \int_a^b \left[ r(x)y^{(n)}(x)^2 - q(x)y(x)^2 \right] dx.
\]

For our basic criterion of oscillation we consider a vector-matrix formulation of \( L(y) = 0 \). Let the \( n \times n \) matrices of functions \( A = [a_{ij}] \), \( B = [b_{ij}] \) and \( C = [c_{ij}] \) be defined by:

\[
a_{ij} = 0 \quad \text{if } j \neq i \neq 1, \quad b_{ij} = 0 \quad \text{if } i \neq n \text{ or } j \neq n
\]

\[
= 1 \quad \text{if } j - i = 1, \quad = \frac{1}{r} \quad \text{if } i = n \text{ and } j = n
\]

and

\[
c_{ij} = 0 \quad \text{if } i \neq 1 \text{ or } j \neq 1, \quad = -q \quad \text{if } i = 1 \text{ and } j = 1.
\]

Then if \( L(y) = 0 \),

\[
\eta = \left[ y^{(i-1)} \right]_{i=1}^n \quad \text{and} \quad \xi = \left[ (-1)^{n-i}(ry^{(n)}(y^{(n-1)})_{i=1}^n,
\]

it is readily verified that

\[
\eta' = A\eta + B\xi, \quad \xi' = C\eta - A^T\xi
\]

where \( A^T \) denotes the transpose of \( A \).

Conversely, if \((\eta, \xi)\) is a pair of absolutely continuous real vector-valued functions on \([a, b]\) such that (7) hold almost everywhere, then it follows that (7) hold everywhere, and the first component \( \eta_1 \) of \( \eta \) is admissible for \( L \) and \( L(\eta_1) = 0 \).

Reid [5, p. 673] has defined the system (7) to be oscillatory on \([a, b]\) if and only if there is a pair \((\eta, \xi)\) of absolutely continuous real or complex vector-valued functions on \([a, b]\) such that (7) hold almost everywhere, \( \eta \neq 0 \), and there are numbers \( c \) and \( d \), \( a \leq c < d \leq b \), such that \( \eta(c) = 0 = \eta(d) \). The one-to-one correspondence between solutions \( y \) of \( L(y) = 0 \) and the first components \( \eta_1 \) of solutions of (7) proves that \( L \) is oscillatory on \([a, b]\) if and only if (7) is oscillatory on \([a, b]\).

We remark that in the terminology of Reid [5, p. 673], the system (7) is identically normal on every subinterval of \([a, \infty)\) since if
(η, ξ) is a solution and η≡0, then the first equation of (7) implies ξ_n≡0, and the second equation of (7) implies successively that ξ_{n-1}≡0, ⋯, ξ_1≡0.

Our basic criterion for oscillation is the following:

**Theorem 1.** If there exists a y∈Ω(b), y≠0, such that I_b(y) ≤ 0, then L is oscillatory on [a, b].

**Proof.** We note first that C(x)r = C(x), B(x)r = B(x) and that B(x) is nonnegative definite on [a, b]. Moreover, if y∈Ω(b), η = ∑_{t=1}^n y^{(t-1)}_t, ξ = ∑_{t=1}^n ξ_t where ξ_n = r y^{(n)} and ξ_t = 0 otherwise, then

$$\xi^T B \xi + \eta^T C \eta = r (y^{(n)})^2 - q y^2.$$  

Then in the terminology of Reid [5, p. 678], we have (η, ξ)∈Ω[a, b], η≠0 and I[η, ξ; a, b] = I_b(y) ≤ 0. Thus by Theorem 5.2 of [5], L is oscillatory on [a, b].

From Theorem 1 we obtain a comparison theorem for oscillation.

**Theorem 2.** If r_1 and q_1 are continuous, real-valued functions on [a, b] with r_1 positive-valued, y∈Ω(b) is a nontrivial solution of L(y) = 0 and L_1(y) = (r_1 y^{(n)})^{(n)} + (-1)^{n-1}q_1 y, then L_1 is oscillatory on [a, b] if

$$\int_a^b [(r(x) - r_1(x)) y^{(n)}(x)^2 - (q(x) - q_1(x)) y(x)^2] dx ≥ 0.$$  

**Proof.** Let J_b be defined by the right-hand side of (5) where r and q are replaced by r_1 and q_1 respectively. Then equation (8) reduces to I_b(y) = J_b(y) ≤ 0. Integrating ∫_a^b (r y^{(n)}(x)) y^{(n)}(x) dx by parts n times proves that I_b(y) = (-1)^n ∫_a^b L(y)·y(x) dx. Since L(y) = 0, equation (8) is equivalent to J_b(y) ≤ 0. Theorem 1 now implies L_1 is oscillatory on [a, b].

As a corollary we have a generalization of the Sturm-Picone Theorem for second-order equations.

**Corollary 2.1.** If L is oscillatory on [a, b] and r_1(x) ≤ r(x) and q_1(x) ≤ q(x) for each x in [a, b], then L_1 is oscillatory on [a, b].

We now prove our principal theorem.

**Theorem 3.** If there is a positive-valued continuous function h on [a, ∞) such that as t→∞,

(i) \( \int_a^t x^{n-1} h(x) dx → ∞ \)

(ii) \( \lim \inf J(t) = -∞ \)

where
\[ J(t) = \left[ \int_a^t \left\{ r(x) [(n-1)!h(x)]^2 - q(x) \left\{ \int_x^t (s-x)^{n-1}h(s)ds \right\}^2 \right\} dx \right] \times \left[ \int_a^t x^{n-1}h(x)dx \right]^{-2}, \]

then there is a number \( b > a \) such that \( L \) is oscillatory on \([a, b]\).

**Proof.** For each number \( t > a + 1 \) we construct a function \( y_t \) on \([a, t]\) such that \( y_t \in \mathcal{A}(t) \). For some \( t \) sufficiently large, we will have \( I_t(y_t) < 0 \), thus proving Theorem 3.

For \( t > a + 1 \), define \( z_t \) on \([a, t]\) by

\[ z_t(x) = \left\{ \int_x^t (s-x)^{n-1}h(s)ds \right\} \left[ \int_a^t x^{n-1}h(s)ds \right]^{-1}. \]

It is clear that for \( k = 0, \ldots, n-1 \),

\[ \left[ d^k z_t(x)/dx^k \right]_{x=a+1} = 0. \]

For \( k = 0, \ldots, n-1 \), let

\[ c_{tk} = \left[ d^k z_t(x)/dx^k \right]_{x=a+1}. \]

Application of l'Hospital's rule proves that \( c_{t0} \to 1 \) and for \( k = 1, \ldots, n-1 \), \( c_{tk} \to 0 \) as \( t \to \infty \).

Let \( p_t \) be the polynomial

\[ p_t(x) = (x - a)^n \sum_{j=0}^{n-1} a_j x^j \]

satisfying, for \( k = 0, \ldots, n-1 \),

\[ \left[ d^k p_t(x)/dx^k \right]_{x=a+1} = c_{tk}. \]

A simple calculation proves that for \( k = 0, \ldots, n-1 \),

\[ \left[ d^k p_t(x)/dx^k \right]_{x=a} = 0. \]

The \( n \) coefficients \( a_0, \ldots, a_{n-1} \) are determined as solutions to the \( n \) linear equations (9). The matrix of coefficients does not depend on \( t \). That the determinant of the matrix of coefficients is nonzero follows from Theorem II of [4] and the fact that \( p_t \) is a solution of the differential equation \( y^{(2n)} = 0 \). Hence \( a_0, \ldots, a_{n-1} \) are bounded functions of \( t \). Define \( y' \) by:

\[ y_t(x) = p_t(x) \text{ for } a \leq x \leq a + 1 \text{ and } y_t(x) = z_t(x) \text{ for } a + 1 < x \leq t. \]

Then \( y_t \in \mathcal{A}(t) \).
We have \( I_t(y_t) = P(t) + K(t) \), where

\[
P(t) = \int_a^{a+1} \left[ r(x) \left( p_t^{(n)}(x) \right)^2 - q(x) p_t(x)^2 \right] dx
\]

and

\[
K(t) = \int_a^t \left[ r(x) \left( z_t^{(n)}(x) \right)^2 - q(x) z_t(x)^2 \right] dx.
\]

Since \( a_0, \ldots, a_{n-1} \) are bounded functions of \( t \), \( P(t) = O(1) \) as \( t \to \infty \). If \( M \) is a bound for \( r, q \) and \( h \) on \([a, a+1]\), then

\[
\left| \int_a^{a+1} \left\{ r(x)[(n-1)!h(x)]^2 - q(x) \left[ \int_x^t (s-x)^{n-1} h(s) ds \right]^2 \right\} dx \right| 
\leq M \left\{ [(n-1)!M]^2 + \left[ \int_a^t (s-a)^{n-1} h(s) ds \right]^2 \right\}.
\]

Hence as \( t \to \infty \),

\[
\int_a^{a+1} \left[ r(z_t^{(n)}(x))^2 - q(x) z_t(x)^2 \right] dx = O(1).
\]

Thus condition (ii) implies that \( \lim \inf K(t) = -\infty \) as \( t \to \infty \). Hence \( \lim \inf I_t(y_t) = -\infty \) as \( t \to \infty \). In particular, \( I_t(y_t) < 0 \) for some sufficiently large \( t \), thus proving Theorem 3.

For \( h=1 \), we have a useful corollary of Theorem 3.

**Corollary 3.1.** If

\[
\limsup_t t^{-2n} \int_a^t r(x) dx < \infty,
\]

and

\[
\lim_t t^{-2n} \int_a^t q(x)(t-x)^{2n} dx = \infty,
\]

then for some \( b > a \), \( L \) is oscillatory on \([a, b]\).

A weaker but more applicable version of Theorem 3 may be stated as follows:

**Theorem 4.** If there is a positive-valued continuous function \( h \) on \([a, \infty)\) such that as \( t \to \infty \)

(i) \( \int_a^t x^{n-1} h(x) dx \to \infty \),

(ii) \( \limsup \left\{ \int_a^t r(x) h(x)^2 dx \right\} \left\{ \int_a^t x^{n-1} h(x) dx \right\}^{-2} < \infty \) and
Proof. Our proof will consist of proving that (i), (ii) and (iii) of Theorem 4 imply (ii) of Theorem 3. First we prove two lemmas.

**Lemma 1.** If $f$ is a continuous, real-valued function on $[a, \infty)$ such that for some integer $p \geq 0$,

$$\lim_{t \to \infty} t^{-p} \int_{a}^{t} f(x)(t - x)^{p} dx = \infty,$$

then for each integer $k > p$,

$$\lim_{t \to \infty} t^{-k} \int_{a}^{t} f(x)(t - x)^{k} dx = \infty.$$

**Proof.** A straightforward inductive proof using l'Hospital's rule is omitted.

**Lemma 2.** Suppose that (i) and (iii) of Theorem 4 hold, and that for $i = 1, \ldots, n$,

$$Q_{i}(t) = \int_{a}^{t} q(x) \left\{ \int_{x}^{t} (s - x)^{n-1} h(s) ds \right\} (t - x)^{i-1} dx.$$

Then for $i = 1, \ldots, n$ and as $t \to \infty$,

$$(10) \quad Q_{i}(t)/t^{i-1} \int_{a}^{t} x^{n-1} h(x) dx \to \infty.$$

**Proof.** For $i = 1$ we have by (iii),

$$\lim_{t \to \infty} Q_{i}'(t)/t^{n-1} h(t) = \lim_{t \to \infty} t^{1-n} \int_{a}^{t} q(x)(t - x)^{n-1} dx = \infty.$$

Hence (10) holds.

Suppose (10) holds for some $i$, $1 \leq i < n$. Let $M$ be a positive number. Since

$$Q_{i+1}(t) = iQ_{i}(t) + \int_{a}^{t} q(x)(t - x)^{n+i-1} h(t) dx,$$

an application of the inductive hypothesis and Lemma 1 yields that for sufficiently large $t$, say $t \geq t_{0}$,

$$Q_{i+1}'(t) \geq Mit^{i-1} \int_{a}^{t} x^{n-1} h(x) dx + Mt^{n+i-1} h(t).$$
Hence for $t > t_0$,

$$Q_{i+1}(t) \geq Q_{i+1}(t_0) + M \left\{ \int_{t_0}^{t} \left[ i s^{i-1} \int_{a}^{s} x^{n-1} h(x) \, dx + s^{n+i-1} h(s) \right] \, ds \right\}$$

$$= Q_{i+1}(t_0) + M \left( \int_{a}^{t} x^{n-1} h(x) \, dx \right) (t^i - t^i_0) + Mt^i \int_{t_0}^{t} x^{n-1} h(x) \, dx.$$

The above inequality implies

$$\liminf_{t \to \infty} \frac{Q_{i+1}(t)}{t^i} \int_{a}^{t} x^{n-1} h(x) \, dx \geq M,$$

from which we conclude that (10) holds for $i+1$.

That (ii) of Theorem 3 is a consequence of (i), (ii) and (iii) of Theorem 4 now follows by applying l'Hospital's rule and Lemma 2 to

$$\lim_{t \to \infty} \left\{ \int_{a}^{t} q(x) \left( \int_{a}^{t} (s-x)^{n-1} h(s) \, ds \right)^2 \, dx \right\} \left\{ \int_{a}^{t} x^{n-1} h(x) \, dx \right\}^{-2}$$

$$= \lim_{t \to \infty} Q_n(t)/t^{n-1} \int_{a}^{t} x^{n-1} h(x) \, dx = \infty.$$

For $h = 1/r$, condition (i) implies (ii), and we obtain the following special case of Theorem 4.

**Corollary 4.1.** If $\int_{a}^{\infty} x^{n-1} r(x)^{-1} \, dx = \infty$ and

$$\lim_{t \to \infty} t^{1-n} \int_{a}^{t} q(x)(t-x)^{n-1} \, dx = \infty,$$

then there is a number $b > a$ such that $L$ is oscillatory on $[a, b]$.

For $n = 1$, Corollary 4.1 gives the sufficient criterion of Leighton [3].

We note that Lemma 1 and Corollary 3.1 imply the previously mentioned result of Wintner for the equation $y'' + qy = 0$. That the condition

$$t^{-2} \int_{a}^{t} q(x)(t-x)^2 \, dx \to \infty \text{ as } t \to \infty$$

is more general than Wintner's condition is shown by the following example.

**Example.** Let $w(t) = t^2 \sin^2 t$ and let $q = w''$. It then follows that

$$\int_{0}^{t} q(x) \, dx = w'(t) = 2t \sin^2 t + t^2 \sin 2t,$$
\( t^{-1} \int_0^t q(x)(t - x) \, dx = t \sin^2 t \)

and

\( t^2 \int_0^t q(x)(t - x)^2 \, dx = t/3 - (1/2 - 1/4t^2) \sin 2t - (\cos 2t)/2t. \)

Hence by Corollary 3.1, \( L(y) = y'' + qy \) is oscillatory.

We remark that Theorem 4 is not applicable in this example since condition (iii) does not hold.

References