SEQUENTIALLY 1-ULC SURFACES IN E^3

C. E. BURGESS[1] AND L. D. LOVELAND

Bing [**4**] has shown that a 2-sphere S in E^3 is tame from the component U of E^3-S if there exists a sequence $\{S_i\}$ of 2-spheres in U converging homeomorphically to S. Hosay [**10**] has announced that a subcontinuum F of S lies on a tame 2-sphere if there exist sequences $\{S_i\}$ and $\{S_i'\}$ of 2-spheres converging homeomorphically to S such that, for each i, $F\subset \operatorname{Int} S_i$ and $F\subset \operatorname{Ext} S_i'$. This theorem was proved independently by Loveland [**11**, Theorems 8, 9, and 10]. A sequence $\{S_i\}$ of 2-spheres that converges homeomorphically to a 2-sphere S must converge 1-regularly; that is, for each $\epsilon>0$ there exists a $\delta>0$ and an integer k such that if $n>k$ and J is a simple closed curve on S_n of diameter less than δ, then some disk on S_n has J as its boundary and is of diameter less than ϵ [**14**, p. 411]. Gillman [**9**] has imposed this property on a sequence $\{T_n\}$ of tori, with the additional requirement that J should be the boundary of a disk on T_n, to obtain a characterization of tame arcs and tame simple closed curves in E^3. Our main purpose in this paper is to impose a slightly weaker property on a sequence of 2-manifolds to obtain conditions implying that a closed subset of a 2-sphere in E^3 lies on a tame 2-sphere (Theorem 3) and to characterize tame 2-spheres in E^3. We then apply Theorem 3 to obtain a generalization (Theorem 4) of Gillman's characterization of tame simple closed curves.

Clearly there exists a sequence $\{S_n\}$ of 2-spheres that converges 1-regularly to a 2-sphere S and does not converge homeomorphically. Roughly speaking, the spheres in such a sequence would have some large folds, but only finitely many of them could have long tentacles. However, the hypotheses of the theorems in this paper permit long tentacles in infinitely many of the manifolds. It is interesting to note that Whyburn has shown that any sequence of 2-spheres that converges 0-regularly to a 2-sphere must converge homeomorphically [**15**, p. 469].

Suppose V is an open set in a 3-manifold and $\{M_i\}$ is a sequence of sets in V. We define $\{M_i\}$ to be *sequentially* 1-ULC *in* V if the following condition is satisfied: For each $\epsilon>0$ there exists a $\delta>0$ and a postive integer k such that if $n>k$ and J is a simple closed curve in M_n of diameter less than δ, then J can be shrunk to a point in an ϵ-subset of V. (A set is called an ϵ-set if its diameter is less than ϵ.)

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Let M be a connected 2-manifold that separates a connected 3-manifold M^3, and let U be a component of M^3-M. We define M to be *locally tame from* U at the point $p\in M$ if there exists a 3-cell K and a disk D such that $K\cap M=D$, $p\in\operatorname{Int} D$, $D\subset\operatorname{Bd} K$, and $K-D\subset U$. The 2-manifold M is tame from U if M is locally tame from U at each point of M. Furthermore, M is tame if it is tame from each component of M^3-M [**1**], [**13**].

We say that U and a subset F of M satisfy *Property* (C, F, U) if for each $\epsilon>0$ there is a $\delta>0$ such that each unknotted simple closed curve that is in U and has diameter less than δ can be shrunk to a point in an ϵ-subset of M^3-F. If Property (C, F, U) is satisfied for each component U of M^3-M, we say that F and M satisfy Property (C, F, M). Properties (A, F, M), (B, F, M), and $(*, F, M)$, as defined in [**11**], are related to Property (C, F, M).

LEMMA 1. *Suppose F is a closed subset of a 2-sphere S in E^3 and U is a component of E^3-S. If for each $p\in F$ and for each open set N containing p there is an open set V containing p such that each unknotted simple closed curve in $V\cap U$ can be shrunk to a point in $N-F$, then Property (C, F, U) is satisfied.*

The above lemma can be proved by using the compactness of F and the existence of a Lebesgue number for an open covering of F [**5**, p. 294].

LEMMA 2. *If F is a closed subset of a 2-sphere S in E^3 such that the diameters of the components of F have a positive lower bound, U is a component of E^3-S, and Property (C, F, U) is satisfied, then F lies on a 2-sphere that is tame from one of its complementary domains.*

Lemma 2 appears as Theorem 2 of [**12**].

LEMMA 3. *If F and S are as in Lemma 2 and Property (C, F, S) is satisfied, then F lies on a tame 2-sphere.*

PROOF. The proof of Theorem 10 of [**11**] also shows that the hypothesis of Lemma 3 implies Property $(*, F, S)$, as defined in [**11**]. Property $(*, F, S)$ is sufficient for F to lie on a tame 2-sphere [**11**, Theorem 6].

THEOREM 1. *Suppose that F is a closed subset of a 2-sphere S in E^3 such that the diameters of the components of F are bounded below by a positive number, U is a component of E^3-S, and $x\in U$. If there exists a sequence $\{M_i\}$ of 2-manifolds such that $\{M_i\cap U\}$ converges to a subset of S, $\{\mathrm{cl}(M_i\cap U)\}$ is sequentially* 1-ULC *in E^3-F, and each M_i sepa-*

rates x from F, then F lies on a 2-sphere that is tame from one of its complementary domains.

PROOF. Let p be a point of F, and let N be an open set containing p. We shall show the existence of an open set H containing p such that each unknotted simple closed curve in $H\cap U$ can be shrunk to a point in $N-F$. Then it will follow from Lemma 1 that Property (C, F, U) is satisfied, and the conclusion of Theorem 1 will follow from Lemma 2.

There exists an open set W containing p such that cl $W\subset N$. We choose a positive number ϵ such that $\epsilon<\rho(W, E^3-N)$. From the hypothesis that $\{\mathrm{cl}(M_i\cap U)\}$ is sequentially 1-ULC in E^3-F, we obtain a $\delta>0$ and an integer k such that if $n>k$ and L is a simple closed curve in $\mathrm{cl}(M_n\cap U)$ of diameter less than δ, then L can be shrunk to a point in an $\epsilon/2$-subset of E^3-F.

Let H' be an open set such that $p\in H'\subset W$ and the diameter of H' is less than $\delta/2$. It follows from the methods in [6] that there exists an open set H containing p such that for each unknotted simple closed curve J in $H\cap U$ there is a Cantor set C in S and a map f of a disk D into H' such that f is a homeomorphism on $D-f^{-1}(C\cap f(D))$, $f(D)$ is locally polyhedral modulo $J\cup C$, $f(\mathrm{Bd}\ D)=J$, $f(D)\cap S\subset C$, and $f(D)\subset S\cup U$. Let J be an unknotted simple closed curve in H and let f be a map satisfying the requirements described above. Since $\{M_i\cap U\}$ converges to a subset of S and each M_i separates F from x, there must exist an integer k' such that M_n separates $J\cup\{x\}$ from F whenever $n>k'$. For the remainder of the proof we assume that $n>\max(k, k')$.

There is a homeomorphism h of M_n such that h is the identity on M_n-U, $h(M_n)\cap U$ is locally polyhedral, and $h(M_n)\cap S=M_n\cap S$ [2]. Furthermore, h can be chosen to move points of M_n so slightly that each $\delta/2$-simple closed curve L in $h(M_n)$ is homotopic in a $\delta/4$-subset of E^3-F to $h^{-1}(L)$. Since diam $h^{-1}(L)<$diam $L+2(\delta/4)<\delta$, each $\delta/2$-simple closed curve in $h(M_n)\cap$cl U can be shrunk to a point in an ϵ-subset of E^3-F.

We assume that $f(D)-S$ and $h(M_n)$ are in relative general position so that $f(D)\cap h(M_n)$ is the union of a null sequence $\{J_i\}$ of simple closed curves in $H\cap(S\cup U)$ and the sets in the sequence $\{J_i\cap U\}$ are disjoint. Thus there exists a null sequence $\{K_i\}$ of sets in E^3 such that, for each i, J_i can be shrunk to a point in K_i. It follows that only finitely many of the K_i intersect F and that, for each of these, the restrictions on H' and δ insure that the corresponding J_i can be shrunk to a point in an ϵ-subset of E^3-F. Thus J can be shrunk to a point in $N-F$.

REMARK. Notice that the proof given for Theorem 1 shows that Property (C, F, U) is satisfied. Thus under the conditions of Corollary 1 we have Property $(C, S, \text{Int } S)$ which is sufficient for S to be tame from Int S (see the proofs given in [5]).

COROLLARY 1. *If S is a 2-sphere in E^3 and $\{S_n\}$ is a sequence of 2-spheres in* Int S *such that* $\{\text{Int } S_n\}$ *converges to* Int S *and* $\{S_n\}$ *is sequentially* 1-ULC *in* Int S, *then* $S \cup \text{Int } S$ *is a 3-cell.*

COROLLARY 2. *If S is a 2-sphere in E^3 and $\{S_n\}$ is a sequence of 2-spheres in* Int S *converging 1-regularly to S such that* $\{\text{Int } S_n\}$ *converges to* Int S, *then* $S \cup \text{Int } S$ *is a 3-cell.*

THEOREM 2. *If the hypothesis of Theorem* 1 *is satisfied with the further requirement that F is a proper subset of S, then F lies on the boundary of a 3-cell.*

PROOF. It follows from Theorem 1 that F lies on a 2-sphere S that is tame from either Int S or Ext S. Of course, if S is tame from Int S then $S \cup \text{Int } S$ is the required 3-cell. If S is tame from Ext S we add a point y to E^3 to obtain a 3-sphere S^3 containing S. Then there is a homeomorphism h of S^3 onto itself such that h is the identity on F and $y \in h(\text{Int } S)$. Now we remove y and $h(S) \cup \text{Int } h(S)$ is the required 3-cell.

THEOREM 3. *If the hypothesis of Theorem* 1 *is satisfied relative to a point x in* Int S *and also relative to a point x' in* Ext S, *then F lies on a tame* 2-sphere.

PROOF. Let $p \in F$, and let N be an open set containing p. Using the proof of Theorem 1 twice, once relative to each component of $E^3 - S$, we obtain an open set V containing p such that if J is an unknotted simple closed curve in $V - S$, then J can be shrunk to a point in $N - F$. Thus Theorem 3 follows from Lemmas 1 and 3.

REMARK. The following question was implicitly raised in both [8] and [11]:

Does Property $(*, F, S)$ hold if F is a closed subset of a 2-sphere S in E^3, F lies on some tame 2-sphere, and F has no degenerate components?

In attempting to obtain a converse of Theorem 3, we have encountered the following related question:

If F is a closed proper subset of a 2-sphere S in E^3 and F lies on some tame 2-sphere, then for each point x in $E^3 - S$ is there a sequence $\{S_i\}$ of 2-spheres converging to a subset of S such that $\{S_i\}$ is sequentially 1-ULC in $E^3 - F$ and each S_i separates x from F?

As observed in [12], the proof of Theorem 10 of [11] shows that (C, F, S) implies $(*, F, S)$ for the case where the diameters of the components of F have a positive lower bound. Thus it follows from the proof of Theorem 1 that an affirmative answer to the latter question would imply, for this special case, an affirmative answer to the former question. Furthermore, it can be shown that the latter question has an affirmative answer if F satisfies $(*, F, S)$ or if F consists of a point. Tame arcs, tame Sierpiński curves, and sets which are their finite union are examples of sets F satisfying $(*, F, S)$ [8], [11].

We modify Gillman's definition [9] of sequentially 1-ULC and say that a sequence $\{T_i\}$ of tori is *weakly sequentially* 1-ULC *in an open set* V if for each $\epsilon>0$ there exists a $\delta>0$ and an integer N such that if $n>N$ and J is a simple closed curve on T_n that bounds a disk on T_n, then J can be shrunk to a point in an ϵ-subset of V.

THEOREM 4. *If a continuum M on a 2-sphere S in E^3 separates S and is the intersection of the interiors of a sequence $\{T_i\}$ of tori such that $\{T_i\}$ is weakly sequentially* 1-ULC *in E^3-M and, for each i, $T_{i+1} \subset \operatorname{Int} T_i$, then M is tame.*

PROOF. By Bing's theorem on polyhedral approximation of surfaces [2], we assume that S is locally polyhedral modulo M and that each T_i is polyhedral and in general position with $S-M$. Let J be a polygonal simple closed curve intersecting both components of E^3-S such that $J\cap S$ is the union of two points a and b which are in different components of $S-M$, and let T be a tubular neighborhood of J such that $\operatorname{cl} T\cap S$ is the union of two disks in $S-M$. For convenience, we assume that each T_i is in $E^3-\operatorname{cl} T$. Let U be a component of E^3-S. We will show that $\{\operatorname{cl}(T_i\cap U)\}$ is sequentially 1-ULC in E^3-M, so this theorem will then follow from Theorem 3. We let ϵ be a positive number and, by hypothesis, we let δ be a positive number and r a positive integer such that, for $n>r$, each δ-simple closed curve that bounds a disk on T_n can be shrunk to a point in an ϵ-subset of E^3-M. We further require that δ be sufficiently small to insure that each simple closed curve that links $\operatorname{cl} T$ has diameter at least δ. Choose $i>r$. There exist two simple closed curves J_1 and J_2 in $T_i\cap S$ such that each of them separates a from b on S. Now let K be a δ-simple closed curve in $\operatorname{cl}(T_i\cap U)$. Since each of J_1 and J_2 links $\operatorname{cl} T$, it follows that neither of them is the boundary of a disk on T_i. Hence there is an annulus A on T_i such that $K\subset A$ and $\operatorname{Bd} A=J_1\cup J_2$. We have required that K be too small to link $\operatorname{cl} T$, so K must be the boundary of a disk in A. Thus K can be shrunk to a point in an ϵ-subset of E^3-M. This shows that $\{\operatorname{cl} T_i\cap U\}$ is sequentially 1-ULC in E^3-M.

The following corollary to Theorem 4 has been proved, with a slightly stronger hypothesis, by Gillman [9].

COROLLARY 3. *If a simple closed curve J on a 2-sphere in E^3 is the intersection of the interiors of a sequence $\{T_i\}$ of tori such that $\{T_i\}$ is weakly sequentially* 1-ULC *in E^3-J and, for each i, $T_{i+1}\subset \operatorname{Int} T_i$, then J is tame.*

THEOREM 5. *Suppose that M is a connected 2-manifold that separates a connected 3-manifold M^3, that V is a component of M^3-M, and that $y\in V$. If there exists a sequence $\{M_i\}$ of 2-manifolds in V converging to M such that $\{M_i\}$ is sequentially* 1-ULC *in V and each M_i separates M from y in M^3, then M is tame from V.*

PROOF. Let $p\in M$, and let Z be an open set containing p such that Z is homeomorphic to E^3. We assume that Z has a triangulation derived from a triangulation of M^3 [3]. It follows from Theorem 5 of [5] that there exists a 2-sphere S and a disk E such that $p\in \operatorname{Int} E \subset E\subset S\cap M$, E is on the boundary of $V\cap \operatorname{Int} S$, and $S\subset Z$. There exists a disk F and a connected open set O such that $p\in \operatorname{Int} F$, $F\subset O \cap M=O\cap S=\operatorname{Int} E$, and $O\cap V=O\cap \operatorname{Int} S$. From the hypothesis that each M_i separates y from M in M^3, it follows that for each point x in $O\cap \operatorname{Int} S$ there exists an integer k such that, for $i>k$, $M_i\cap O\cap V$ separates x from F in O. Thus Property $(C, F, \operatorname{Int} S)$ follows from a slight modification of the proof of Theorem 1. Since the proof of Theorem 6 in [5] is valid with a restriction to unknotted simpled closed curves, $(C, F, \operatorname{Int} S)$ implies that S is locally tame from $\operatorname{Int} S$ at p. Thus M is tame from V [1], [13].

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THE UNIVERSITY OF UTAH AND
UTAH STATE UNIVERSITY