ON FINITE GROUPS WHOSE $p$-SYLOW SUBGROUP IS A T. I. SET

HENRY S. LEONARD, JR.

Throughout this note we let $p$ be a fixed prime and let $G$ be a finite group whose fixed $p$-Sylow subgroup $P$ is a T. I. set (trivial intersection set). That is, the intersection of any two distinct conjugates of $P$ is $\{1\}$. Denote $|P|$ by $p^a$. It is conjectured that if $G$ has a faithful complex character $\chi$ with $\chi(1) \leq p^{a/2} - 1$, then $P \triangleleft G$. This has been confirmed in certain cases [4, page 287 and Lemma 4.2], [6, Theorem 4.3]. In fact under certain conditions it is sufficient to assume $\chi(1) < (p^a - 1)/2$ [1, Theorem 3], [6, Theorem 4.2], but in general the conclusion $P \triangleleft G$ does not hold under this weaker assumption because of the presence of Suzuki's simple groups.

Our purpose here is to use Brauer's theory of the correspondence between $p$-blocks of a subgroup of $G$ and $p$-blocks of $G$ [2], [3], together with a result of Gorenstein and Walter [5, Equation (46)], to obtain the theorem below which verifies the conjecture in the case that $C(V) \subseteq N(P)$, where $V$ is the group of $p$-regular elements of $C(P)$. In particular for any counterexample of minimal order of the conjecture, we would have $C(P) \subseteq PZ(G)$.

The notation is standard. If $H$ is a subgroup of $G$, then $N(H)$, $C(H)$, and $Z(H)$ denote the normalizer, centralizer, and center of $H$. Denote $Z(G)$ by $Z$. All characters are over the complex field.

Assuming $P$ is a T. I. set, let $B$ be a $p$-block of $G$ of defect $\neq 0$, and let $D$ be a defect group of $B$ with $D \subseteq P$ and with $|D| = p^a$. Then $N(D) \subseteq N(P)$ and the $p$-Sylow group of $N(D)$ is normal in $N(D)$. Furthermore by [2, Theorem (8C)], there is a block $\bar{B}$ of $N(D)$ which corresponds to $B$ in the sense of [2, Theorem (7E)]. The defect group of $\bar{B}$ is the $p$-Sylow group of $N(D)$ [2, Theorem (9F)] and is contained in $D$ [2, Theorem (8D)]. We must have $D = P$. Thus every $p$-block of $G$ has defect 0 or full defect $a$.

We know that

\[(1) \quad PC(P) = P \times V\]

where $V$ is a group of order prime to $p$. Then every $p$-block of $PC(P)$ consists of the $p^a$ irreducible characters $\lambda \theta$ where $\theta$ is a fixed irreducible...
character of $V$ and $\lambda$ ranges over all the irreducible characters of $P$. We shall denote this block by $b(\theta)$.

There is a one-to-one correspondence between the $p$-blocks of defect $a$ of $G$ and the classes $\{\theta\}$ of irreducible characters of $V$ associated in $N(P)$ [2, Theorem (12A)]. Denote the block of $G$ corresponding to $\{\theta\}$ by $B(\theta)$. Then, according to [3, Theorem (2D)],

$$b(\theta)^G = B(\theta)$$

in the sense defined there. Every $p$-block of $N(P)$ is of defect $a$ and must be of the form $b(\theta)^N(P)$ for some $\theta$. We denote this block by $\tilde{B}(\theta)$. Then [3, Theorem (2C)] implies

$$\tilde{B}(\theta)^G = B(\theta).$$

**Lemma 1.** An irreducible character $\psi$ of $N(P)$ belongs to $\tilde{B}(\theta)$ if and only if $\psi|_V$ has $\theta$ as a constituent.

**Proof.** Let $\Omega$ be an algebraic number field of finite degree containing the $|N(P)|$ th roots of unity. Let $\mathfrak{o}$ be the ring of algebraic integers in $\Omega$ and let $p$ be a prime ideal of $\mathfrak{o}$ containing $p$.

If we apply (2) to $A(P)$, it follows from [2, Equation (12.2)] that for $\psi \in \tilde{B}(\theta)$ and $v \in V$ we have

$$\left| \frac{N(P)}{C(v) \cap N(P)} \right| \frac{\psi(v)}{\psi(1)} = \sum_{w} \theta(w) \quad (\text{mod } p).$$

Here $w$ ranges over the elements of $V$ which are conjugate to $v$ in $N(P)$. Hence

$$\left| \frac{N(P)}{C(v) \cap N(P)} \right| \frac{\psi(v)}{\psi(1)} = \left| \frac{N(P)}{C(v) \cap N(P)} \right| \frac{1}{q} \sum_{i} \theta_i(v) \quad (\text{mod } p),$$

where $\theta_i$ ranges over the associates of $\theta$ in $N(P)$ and $q$ is the number of these associates. But, since $V \triangle N(P)$,

$$\psi|_V = \frac{\psi(1)}{q' \theta'(1)} \sum_{\{\theta'\}} \theta'_j$$

for some class $\{\theta'\}$ where $q'$ is the number of members of this class. These last two relations yield a congruence relating the values of $\theta$ and its associates to those of $\theta'$ and its associates. However, the irreducible characters of $V$ are linearly independent (mod $p$) [2, Theorem (3C)]. Therefore $\theta$ and $\theta'$ are associates in $N(P)$, and the lemma follows from (4).

Let $D$ denote the set of $p$-singular elements of $G$ whose $p$-factor is
in the fixed $p$-Sylow subgroup $P$. Let $B$ be a $p$-block of $G$, and let $\chi_i \in B$. Then

\[ \chi_i|N(P) = \sum_j a_{ij}\psi_j \]

where the $\psi_j$ are the irreducible characters of $N(P)$ and the $a_{ij}$ are integers. Then according to [5, Equation (46)],

\[ \chi_i|B = \sum' a_{ij}\psi_j|B \]

where we have summed only those terms for which $\psi_j \in \bar{B}$ and $\bar{B}^g = B$ for some block $\bar{B}$ of $N(P)$.

**Lemma 2.** If $\chi_i \in B(\theta)$ and $\chi_i(1) \leq p^a$, then every constituent of $\chi_i|V$ is an associate in $N(P)$ of $\theta$. In particular, if $\theta = 1$, then the kernel of $\chi_i$ contains $V$.

**Proof.** For $\chi_i$ we have an equation of the form (5). It follows from (3) and (6) that

\[ \psi = \sum_{\psi_j \in B(\theta)} a_{ij}\psi_j \]

vanishes on $P - \{1\}$. Hence $\psi|P$ must be a multiple of the character of the regular representation of $P$ so $p^a|\psi(1)$. Since $\chi_i$ is not of defect 0, $\chi_i(1) < p^a$. Hence $\psi$ is identically zero, and (5) and (6) have the same terms. The lemma now follows from Lemma 1.

In particular, $B(1_V)$ is the principal block (containing the principal character $1_G$ of $G$).

**Proposition.** Suppose the $p$-Sylow subgroup $P$ is a T. I. set. If $G$ has an irreducible character $\chi$ such that $\chi|V$ is reducible and $\chi(1) \leq (p^a+1)^{1/2}$, then $G$ has a normal subgroup $M \not\triangleq G$ containing $V$.

**Proof.** It follows from Lemma 2 that $\chi \bar{\chi}$ has a nonprincipal constituent in $B(1)$ and that this constituent has $V$ in its kernel.

**Remark.** If $G$ has a nonprincipal character $\chi$ such that $\chi|V$ is irreducible then without use of the lemmas we see easily that $G$ has a normal subgroup $M \not\triangleq G$ containing either $P$ or $V$.

**Theorem.** Suppose the $p$-Sylow subgroup $P$ of $G$ is a T. I. set and that $C(V) \subseteq N(P)$. If $G$ has a faithful character $\chi$ all of whose constituents have degrees $\leq (p^a+1)^{1/2}$, then $P \triangle G$.

**Proof.** Suppose the statement is false and that $G$ is a counterex-
ample of minimal order. If for every constituent $\chi_0$ of $\chi$, $\chi_0|_{PV}$ is irreducible then $Z(P) \subseteq Z(G)$ and $P \triangle G$, which is not the case. Hence for some constituent $\chi_0$ of $\chi$, $\chi_0|_{PV}$ is reducible. Then $\chi_0\bar{x}_0$ has a constituent $\chi_1 \neq 1$ such that $1_{PV} \subseteq \chi_1|_{PV}$. By Lemma 2, $V \subseteq K$, the kernel of $\chi_1$. Either $KN(P) = G$ or $P \triangle KN(P)$. In the first case, $\chi_1|_{N(P)}$ is irreducible and then $P \subseteq K \triangle G$. By the minimality of $G$, $P \triangle K \triangle G$, which is not the case.

Thus $P \triangle KN(P)$. Then $K \cap P = 1$ since $P \triangle G$. Hence $KP = K \times P$, so $V \subseteq K \subseteq V$ and $V \triangle G$. Then $P \triangle VC(V) \triangle G$ so $P \triangle G$. This is a contradiction and the proof is complete.

References

1. R. Brauer, On groups whose order contains a prime number to the first power. II, Amer. J. Math. 64 (1942), 421-440.

Carnegie Institute of Technology