RINGS OF CONTINUOUS FUNCTIONS ON OPEN CONVEX SUBSETS OF R^n

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0. **Introduction.** If h is a homeomorphism from a topological space X onto another space Y, then one can easily show that the correspondence $f \to f \circ h^{-1}$ is an isomorphism from $C(X)$, the ring of all real continuous functions on X, onto $C(Y)$, the ring of all real continuous functions on Y. The converse problem—given an isomorphism ϕ from $C(X)$ onto $C(Y)$ to show that there exists a homeomorphism h from X onto Y such that $\phi f = f \circ h^{-1}$ —is more difficult and is one of the problems which has motivated much of the research on rings of continuous functions. The isomorphism ϕ is usually constructed by using *fixed*, maximal ideals. An ideal M in $C(X)$ is fixed if there exists a point x_0 of X such that $f(x_0) = 0$ for all f in M. If X is completely regular, then the set $M(x)$ of all functions in $C(X)$ which are zero at x is a fixed, maximal ideal and all fixed, maximal ideals are of this form. If one can show that the property of a maximal ideal being fixed is invariant under ring isomorphisms and if X and Y are completely regular, then one can show that the correspondence

$$x \to M(x) \to \phi(M(x)) = M(y) \to y,$$

where $M(y)$ is the set of all functions in $C(Y)$ which are zero at a point y of Y, is the desired homeomorphism h. If X and Y are compact, then all maximal ideals are fixed, but if X and Y are not compact, then there are maximal ideals in $C(X)$ and $C(Y)$ which are not fixed. (Nonfixed ideals are called "free" ideals.) The maximal, fixed ideals in $C(X)$ have been algebraically characterized for certain classes of noncompact spaces, X, including the class of all separable metric spaces and the class of all normal Hausdorff spaces whose points are G-delta sets, but these characterizations are complicated and/or difficult to establish. (References: Gelfand and Kolmogoroff [**1**], Hewitt [**4**], Pursell [**5**], and Gillman and Jerison [**2**].)

In this paper we give a construction of the homeomorphism h for the case X and Y are open convex subsets of R^n which does not depend on an algebraic characterization of fixed, maximal ideals. This method of construction will also give us the theorem:

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If $C^\infty(X)$ and $C^\infty(Y)$ are the rings of all real, infinitely differentiable functions on the open convex subsets X and Y of R^n, respectively and ϕ is an isomorphism from $C^\infty(X)$ onto $C^\infty(Y)$, then there is a diffeomorphism h from X onto Y such that $\phi f = f \circ h^{-1}$ for all f in $C^\infty(X)$.

Our construction depends upon two properties of infinitely differentiable functions on open convex subsets X of R^n:

(0.1) A function in $C(X)$ can be uniformly approximated on X by a function in $C^\infty(X)$.

(0.2) If f is in $C^\infty(X)$, then to each point $\bar{a} = (a_1, a_2, \cdots, a_n)$ in X, there corresponds an n-tuple $(f_{\bar{a},1}, f_{\bar{a},2}, \cdots, f_{\bar{a},n})$ of functions in $C^\infty(X)$ such that

$$f(\bar{r}) = f(\bar{a}) + \sum_{i=1}^{n} (r_i - a_i) f_{\bar{a},i}(\bar{r})$$

for all $\bar{r} = (r_1, r_2, \cdots, r_n)$ in X.

The proof of the first of these properties is given in detail in [**6**] for $X = R$ and is outlined there for X any paracompact differentiable manifold. The second is proved in [**3**, pp. 9–10]. It is a simple consequence of a theorem concerning the differentiability of the remainder term in Taylor's formula which was proved earlier for R and R^2 by H. Whitney [**7**]. (See also [**8**].)

1. **Preliminaries.** In the sequel X and Y always denote open convex subsets of R^n. We denote n-tuples of real numbers or real functions by a bar over a symbol. For example, we denote arbitrary points in R^n by $\bar{r} = (r_1, r_2, \cdots, r_n)$ and arbitrary elements in $(C(X))^n$, that is continuous mappings from X into R^n, by $\bar{f} = (f_1, f_2, \cdots, f_n)$ or $\bar{g} = (g_1, g_2, \cdots, g_n)$. If $\bar{f}$ is in $(C(X))^n$ and ϕ is an isomorphism from $C(X)$ to $C(Y)$, then by $\phi\bar{f}$ we mean the n-tuple $(\phi f_1, \phi f_2, \cdots, \phi f_n)$ in $(C(Y))^n$. The projection mappings or coordinates on X are denoted by $x_1, x_2, \cdots, x_n$. That is $x_i(\bar{r}) = r_i$ for all $\bar{r}$ in X. Hence $\bar{x}$ is the identity mapping on X and $\bar{y} = (y_1, y_2, \cdots, y_n)$ is the identity mapping on Y. We denote constant functions by boldface. For example if a is a real number, then $\boldsymbol{a}_X(\bar{r}) = a$ for all $\bar{r}$ in X and $\boldsymbol{a}_Y(\bar{s}) = a$ for all $\bar{s}$ in Y. Observe that $\mathbf{1}_X$ is the multiplicative identity in $C(X)$. The set of all real constants on X is denoted by $\boldsymbol{R}(X)$ and on Y by $\boldsymbol{R}(Y)$.

2. **Construction of the homeomorphism.** Our main theorem may now be stated:

(2.1) THEOREM. *If X and Y are open convex subsets of R^n and ϕ is an isomorphism from $C(X)$ onto $C(Y)$, then $\phi\bar{x}$ is a homeomorphism from Y onto X and $\phi f = f(\phi\bar{x})$ for all f in $C(X)$.*

Our proof requires several lemmas.

(2.2) LEMMA. *If ϕ is any isomorphism from $C(X)$ onto $C(Y)$, then $\phi \boldsymbol{a}_X = \boldsymbol{a}_Y$ for all $\boldsymbol{a}_X$ in $\boldsymbol{R}(X)$.*

PROOF. For each point $\bar{r}$ in Y, the correspondence $a \rightarrow (\phi \boldsymbol{a}_X)(\bar{r})$ is a homomorphism $\psi_{\bar{r}}$ from R into R. Hence $\psi_{\bar{r}}$ is either the zero homomorphism or the identity mapping on R. But since $\mathbf{1}_X$ is the multiplicative identity in $C(X)$, then $\phi \boldsymbol{1}_X = \boldsymbol{1}_Y$ and, therefore, $\psi_{\bar{r}} 1 = (\phi \boldsymbol{1}_X)(\bar{r}) = 1$ for all $\bar{r}$. Therefore $\psi_{\bar{r}}$ must be the identity mapping on R for all $\bar{r}$ in Y (compare [**2**, p. 23]).

(2.3) LEMMA. *If ϕ is an isomorphism from $C(X)$ onto $C(Y)$ and $\bar{g}$ is in $(C(X))^n$, then $(\phi\bar{g})(Y) = \bar{g}(X)$.*

PROOF. Let $\bar{a}$ be any point of R^n and define

$$f_{\bar{a}} = \sum_{i=1}^{n} (g_i - \boldsymbol{a}_{iX})^2.$$

then $f_{\bar{a}}$ is in $C(X)$ and has a multiplicative inverse in $C(X)$ if and only if $\bar{a}$ is in $R^n - \bar{g}(X)$. Hence $\phi f_{\bar{a}}$ has a multiplicative inverse in $C(Y)$ if and only if $\bar{a}$ is in $R^n - \bar{g}(X)$. But $\phi f_{\bar{a}} = \sum_{i=1}^{n} (\phi g_i - \boldsymbol{a}_{iY})^2$ has a multiplicative inverse in $C(Y)$ if and only if $\bar{a}$ is in $R^n - (\phi\bar{g})(Y)$.

(2.4) LEMMA. *If ϕ is an isomorphism from $C(X)$ onto $C(Y)$, then $\phi(f \circ \bar{g}) = f(\phi\bar{g})$ for all f in $C^\infty(X)$ and $\bar{g}$ in $(C(X))^n$ such that $\bar{g}$ maps X into X.*

PROOF. By (0.2)

$$f(\bar{r}) = f(\bar{a}) + \sum_i (r_i - a_i) f_{\bar{a},i}(\bar{r})$$

for all $\bar{a}$, $\bar{r}$ in X. Substituting $\bar{g}$ for $\bar{r}$, we obtain

$$f \circ \bar{g} = \boldsymbol{f}(\bar{\boldsymbol{a}})_X + \sum_i (g_i - \boldsymbol{a}_{iX}) f_{\bar{a},i} \circ \bar{g}.$$

Hence by Lemma (2.2)

$$(\phi(f \circ \bar{g}))(\bar{s}) = f(\bar{a}) + \sum_i [(\phi g_i)(\bar{s}) - a_i] \cdot [(\phi(f_{\bar{a},i} \circ \bar{g}))(\bar{s})]$$

for all $\bar{a}$ in X and $\bar{s}$ in Y. Since $(\phi\bar{g})(\bar{s})$ is in X by (2.3), then we may set $\bar{a} = (\phi\bar{g})(\bar{s})$, obtaining $(\phi(f \circ \bar{g}))(\bar{s}) = f((\phi\bar{g})(\bar{s}))$ for all $\bar{s}$ in Y.

We now use (0.1) to extend (2.4) to all functions f in $C(X)$.

(2.5) LEMMA. *If ϕ is an isomorphism from $C(X)$ onto $C(Y)$ and $\bar{g}$ in $(C(X))^n$ maps X into X, then $\phi(f \circ \bar{g}) = f(\phi\bar{g})$ for all f in $C(X)$.*

PROOF. By (0.1), for each f in $C(X)$ and each $\epsilon > 0$, there is a corresponding f_ϵ in $C^\infty(X)$ such that

$$f_\epsilon(\bar{r}) - \epsilon \leqq f(\bar{r}) \leqq f_\epsilon(\bar{r}) + \epsilon \qquad \text{for all} \quad \bar{r} \quad \text{in} \quad X. \tag{2.5.1}$$

Hence

$$(f_\epsilon \circ \bar{g}) - \epsilon_X \leqq f \circ \bar{g} \leqq (f_\epsilon \circ \bar{g}) + \epsilon_X.$$

Since a function in $C(X)$ is nonnegative on X if and only if it has a square root in $C(X)$, the isomorphism ϕ is order-preserving. Therefore by (2.4)

$$\begin{aligned} f_\epsilon(\phi\bar{g}) - \epsilon_Y = \phi[(f_\epsilon \circ \bar{g}) - \epsilon_X] \leqq \phi(f \circ \bar{g}) \leqq \phi[(f_\epsilon \circ \bar{g}) + \epsilon_X] \\ = f_\epsilon(\phi\bar{g}) + \epsilon_Y \qquad \text{for all } \epsilon > 0. \end{aligned} \tag{2.5.2}$$

But since $(\phi\bar{g})(Y) = \bar{g}(X) \subseteq X$, we may substitute $\phi\bar{g}$ for $\bar{r}$ in (2.5.1) to obtain

$$f_\epsilon(\phi\bar{g}) - \epsilon_Y \leqq f(\phi\bar{g}) \leqq f_\epsilon(\phi\bar{g}) + \epsilon_Y \qquad \text{for all } \epsilon > 0. \tag{2.5.3}$$

Comparing (2.5.2) and (2.5.3) we have the desired result.

(2.6) COROLLARY. *If ϕ is an isomorphism from $C(X)$ onto $C(Y)$, $\bar{f}$ is a continuous map from X into R^n, and $\bar{g}$ is a continuous map from X into X, then $\phi(\bar{f} \circ \bar{g}) = \bar{f}(\phi\bar{g})$.*

That $\phi f = f(\phi\bar{x})$ is now a special case of (2.5). To complete the proof of Theorem 2.1, we show that $\phi\bar{x}$ is a homeomorphism from Y onto X. Clearly $\phi\bar{x}$ is a continuous map from Y into R^n. By (2.3) $\phi\bar{x}$ maps Y onto X. If we interchange Y and X in (2.3) and replace ϕ by ϕ^{-1}, then we see that $\phi^{-1}\bar{y}$ maps X onto Y. If we substitute $\phi^{-1}\bar{y}$ for $\bar{f}$ and $\bar{x}$ for $\bar{g}$ in (2.6), we obtain

$$\phi((\phi^{-1}\bar{y}) \circ \bar{x}) = (\phi^{-1}\bar{y}) \circ (\phi\bar{x}).$$

But since $\bar{x}$ is the identity map on X,

$$\phi((\phi^{-1}\bar{y}) \circ \bar{x}) = \phi(\phi^{-1}\bar{y}) = \bar{y}.$$

Hence $\phi\bar{x}$ has a continuous inverse $\phi^{-1}\bar{y}$. Therefore $\phi\bar{x}$ is a homeomorphism from Y onto X and Theorem 2.1 is proved.

If $Y = X$ and we replace ϕ^{-1} in the above discussion by an arbitrary automorphism on $C(X)$, we obtain $\phi(\psi\bar{x}) = (\psi\bar{x}) \circ (\phi\bar{x})$. It follows that the correspondence $\phi \rightarrow \phi\bar{x}$ is an anti-isomorphism from the group of all automorphisms on $C(X)$ onto the group of all homeomorphisms on X.

In Lemmas (2.2), (2.3), (2.4), and (2.6) we could have replaced the rings $C(X)$ and $C(Y)$ by the rings $C^\infty(X)$ and $C^\infty(Y)$, obtaining:

(2.7) LEMMA. *If ϕ is an isomorphism from $C^\infty(X)$ onto $C^\infty(Y)$, $\bar{f}$ and $\bar{g}$ are in $(C^\infty(X))^n$, and $\bar{g}$ maps X into X, then*

(i) $(\phi\bar{f})(Y)=\bar{f}(X)$,

(ii) $\phi(\bar{f}\circ\bar{g})=\bar{f}(\phi\bar{g})$,

and for any function f in $C^\infty(X)$

(iii) $\phi f=f(\phi\bar{x})$.

That $\phi\bar{x}$ is a diffeomorphism if ϕ is an isomorphism from $C^\infty(X)$ onto $C^\infty(Y)$ may be proved in the same way that we proved above that $\phi\bar{x}$ is a homeomorphism if ϕ is an isomorphism from $C(X)$ onto $C(Y)$. Hence:

(2.8) THEOREM. *If X and Y are open convex subsets of R^n and ϕ is an isomorphism from $C^\infty(X)$ onto $C^\infty(Y)$, then $\phi\bar{x}$ is a diffeomorphism from Y onto X and $\phi f=f(\phi\bar{x})$ for all f in $C(X)$.*

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