MEASURES WITH SEPARABLE ORBITS

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1. Introduction. Let \( G \) be a locally compact topological (LC) group, and denote by \( V(G) \) the linear space of all complex-valued regular Borel measures on \( G \) which are finite on compact sets. For \( \mu \in V(G) \) and \( s \in G \) let \( T_s\mu \) be that measure in \( V(G) \) defined by \( T_s\mu(E) = \mu(ES) \) for each Borel set \( E \) with compact closure. We shall denote by \( m \) and \( dm \) right invariant Haar measure on \( G \), i.e., \( T_s m = m(s \in G) \); and by \( |\mu| \) the total variation of the measure \( \mu \). A measure \( \mu \in V(G) \) is said to have a separable orbit if there exists a countable subset \( C \subseteq G \) with the property that for each \( s \in G \) and \( \epsilon > 0 \) there exists some \( c \in C \) such that \( |T_s\mu - T_c\mu| (G) = \|T_s\mu - T_c\mu\| \leq \epsilon \).

The main result of this paper is the theorem which asserts for second countable LC groups \( G \) that if \( \mu \in V(G) \) has a separable orbit then \( \mu \) is absolutely continuous with respect to \( m \). In the case that \( |m| (G) < \infty \) an immediate consequence of this result is the equivalence of the absolute continuity of \( \mu \) and the separability of the orbit of \( \mu \).

As usual \( M(G) \) will denote the space of \( \mu \in V(G) \) for which \( |\mu| (G) < \infty \).

2. Lemmas.

Lemma 1. Let \( G \) be a LC group and \( \mu \in V(G) \). For each Borel set \( E \subseteq G \) the complex-valued function \( T_\mu(E) \) is Borel measurable.

Proof. Let \( \chi_E \) denote the characteristic function of the set \( E \). Then \( \chi_E(ts^{-1}) \) is a Borel measurable function on \( G \times G \) and hence \( \int_G \chi_E(ts^{-1}) d\mu(t) = T_s\mu(E) \) is a Borel measurable function on \( G \).

Lemma 2. Let \( G \) be a second countable LC group, and suppose \( \mathcal{R} \) is the ring of sets generated by some countable basis for \( G \) consisting of compact neighborhoods. If \( \mu \in V(G) \) and \( K \subseteq G \) is any compact neighborhood then \( |\mu| (K) = \sup \left\{ \sum_i |\mu(L_i)| : \text{the supremum is taken over all finite collections of pairwise disjoint sets } L_i \in \mathcal{R} \text{ whose union is contained in } K \right\} \).

Proof. It is evident that \( \sup \sum_i |\mu(L_i)| \leq |\mu| (K) \).

Conversely, suppose \( \epsilon > 0 \) and let \( \mu \) be a bounded Borel measurable function such that \( |g| = 1 \) and \( d\mu = gd|\mu| \) [1, p. 171]. First, using the regularity of \( \mu \), we choose a compact set \( C \subseteq K \) such that \( |\mu| (K - C) < \epsilon/3 \). Then since \( K \) is open we may apply Lusin's theorem [3, p. 53].

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to find a continuous complex-valued function $f$ with compact support $F$ such that (i) $C \subset F \subset K$, (ii) $\sup |f(t)| \leq \sup |\chi_c(t)/g(t)| = 1$, and (iii) $|\mu| \left( \{ t | f(t) \neq \chi_c(t)/g(t) \} \right) < \varepsilon/6$. Finally, employing the openness of $K$, the regularity of $\mu$ and a standard construction for the approximation of continuous functions, we construct a function of the form $h = \sum_{i=1}^{n} c_i \chi_{L_i}$, where $|c_i| \leq 1$, $L_i \in \mathcal{M}$, $i = 1, 2, \cdots, n$, $L_i \cap L_j = \emptyset$, $i \neq j$, $\bigcup_{i=1}^{n} L_i \subset K$, and such that $\int_{G} |f-h| \, d|\mu| < \varepsilon/3$.

Combining these inequalities we conclude that,

$$\left| \int_{G} \chi_{Kd} \, |\mu| - \int_{G} hgd \, |\mu| \right| \leq \int_{G} |\chi_{K}/g - h| \, d\mu$$

$$\leq \int_{G} |\chi_{K}/g - \chi_{c}/g| \, d\mu + \int_{G} |\chi_{c}/g - f| \, d\mu$$

$$+ \int_{G} |f - h| \, d\mu < \varepsilon/3 + 2\varepsilon/6 + \varepsilon/3 = \varepsilon;$$

and hence

$$|\mu|(K) - \varepsilon < \left| \int_{G} hgd \, |\mu| \right| = \left| \sum_{i=1}^{n} c_i \mu(L_i) \right| \leq \sum_{i=1}^{n} |\mu(L_i)|.$$

Therefore $|\mu|(K) = \sup \sum_i \mu(L_i)$.

The third lemma follows immediately from the previous two lemmas.

**Lemma 3.** Let $G$ be a second countable LC group and $\mu \in V(G)$. For each compact neighborhood $K \subset G$ the real-valued function $|T_s \mu - \mu|(K)$ is Borel measurable.

3. **Theorems.** To establish the theorem mentioned in the introduction we shall need, besides Lemma 3, one further result which we state without proof. The proof given in [2, p. 230] is for abelian groups and finite measures but it is apparent that with some minor modifications of the argument the theorem given here is also valid.

**Theorem 1.** Let $G$ be a LC group and $\mu \in V(G)$. Then the following are equivalent:

(i) $\mu$ is absolutely continuous with respect to $m$.

(ii) For each compact $K \subset G$ the function $T_s |\mu|(K)$ is continuous.

The main result of the paper is the following theorem.

**Theorem 2.** Let $G$ be a second countable LC group and $\mu \in V(G)$. If $\mu$ has a separable orbit then $\mu$ is absolutely continuous with respect to $m$. 
Proof. We note first that the function \(|T_s\mu - \mu|\) is a Borel measurable function on \(G\). Indeed, since \(G\) is \(\sigma\)-compact there exists a sequence of compact neighborhoods \(K_i\) such that \(K_i \subseteq K_{i+1}\) and \(\bigcup_{n=1}^{\infty} K_i = G\). The measurability of \(|T_s\mu - \mu|\) is then a consequence of Lemma 3 and the fact that for each \(s \in G\) \(|T_s\mu - \mu| = \lim_{B} |T_s - \mu| (B)\).

Let \(\epsilon > 0\) and set \(A = \{s \mid |T_s\mu - \mu| \leq \epsilon/2\}\). It is evident that \(A\) is nonempty, symmetric and, as previously remarked, measurable. Moreover \(A\) has positive Haar measure. For \(s \in G\) choose \(c \in C\) such that \(|T_c\mu - T_s\mu| \leq \epsilon/2\). A \(c\) satisfying this property exists since the orbit of \(\mu\) is separable. However, \(|T_c^{-1}s\mu - \mu| = |T_c^{-1}s\mu - T_c^{-1}s\mu - T_c^{-1}s\mu| \leq \epsilon/2\), and so \(c^{-1}s \in A\). Thus \(G = \bigcup_{c \in C} cA\), and it follows immediately that \(m(A) > 0\).

Combining these facts we conclude via a well-known theorem [1, p. 296] that \(AA\) contains an open neighborhood of the identity in \(G\). Furthermore an application of the triangle inequality reveals that \(AA \subseteq \{s \mid |T_s\mu - \mu| \leq \epsilon\}\), and hence \(|T_s\mu - \mu|\) is continuous at the identity.

Therefore for each compact \(K \subseteq G\) the function \(|T_s\mu - \mu| (K)\), and consequently also the function \(T_s|\mu| (K)\), is continuous at the identity and thus at all points of \(G\).

Finally, an application of Theorem 1 allows us to conclude that \(|\mu|\), and so \(\mu\), is absolutely continuous with respect to \(m\).

For \(\mu \in M(G)\) the converse of Theorem 2 can be easily proved.

Theorem 3. Let \(G\) be a second countable LC group and \(\mu \in M(G)\). Then the following are equivalent:

(i) \(\mu\) is absolutely continuous with respect to \(m\).

(ii) \(\mu\) has a separable orbit.

Proof. The content of Theorem 2 is that (ii) implies (i).

Suppose \(d\mu = f \, dm\) where \(f\) is some Borel measurable function such that \(\int_G |f| \, dm < \infty\). Then the separability of the orbit of \(\mu\) follows immediately from the continuity of translation of absolutely integrable functions [1, p. 285] and the second countability of \(G\).

Remarks. (a) If the group \(G\) is not second countable then there may exist \(\mu \in M(G)\) which is absolutely continuous with respect to \(m\) but which does not have a separable orbit. An obvious example is the group \(G = R_d\), the additive group of the real numbers with the discrete topology, and \(\mu = \delta\), the unit mass concentrated at zero.

(b) When \(G\) is second countable and \(\mu \in V(G) \sim M(G)\) then it is possible for \(\mu\) to be absolutely continuous but not have a separable orbit. If \(G = R\), the additive group of the real line with the usual topol-
ogy, then \( d\mu(t) = e'^dt \) is such a measure. Thus the converse to Theorem 2 is not valid.

(c) On the other hand there do exist absolutely continuous measures \( \mu \in V(G) \sim M(G) \), other than complex multiples of Haar measure, which have separable orbits. If we again let \( G = \mathbb{R} \) then \( d\mu(t) = f(t)dt \) where \( f(t) = 1, |t| < 1 \), and \( f(t) = 1/|t|, |t| \geq 1 \), is one instance of such a measure.

(d) Because of the form in which Theorem 1 is stated here and in [2] it may be of some interest to note an equivalent formulation of the notion of separability for \( \sigma \)-compact groups, namely: there exists a countable set \( C \subseteq G \) such that for each \( s \in G \) and \( \epsilon > 0 \) there exists some \( c \in C \) such that \( |T_s\mu - T_c\mu|(K) \leq \epsilon \) for all compact sets \( K \subseteq G \).

References