MEASURES WITH SEPARABLE ORBITS

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1. **Introduction.** Let G be a locally compact topological (LC) group, and denote by $V(G)$ the linear space of all complex-valued regular Borel measures on G which are finite on compact sets. For $\mu \in V(G)$ and $s \in G$ let $T_s\mu$ be that measure in $V(G)$ defined by $T_s\mu(E) = \mu(Es)$ for each Borel set E with compact closure. We shall denote by m and dm right invariant Haar measure on G, i.e., $T_s m = m (s \in G)$; and by $|\mu|$ the total variation of the measure μ. A measure $\mu \in V(G)$ is said to have a *separable orbit* if there exists a countable subset $C \subset G$ with the property that for each $s \in G$ and $\epsilon > 0$ there exists some $c \in C$ such that $|T_s\mu - T_c\mu|(G) = \|T_s\mu - T_c\mu\| \leqq \epsilon$.

The main result of this paper is the theorem which asserts for second countable LC groups G that if $\mu \in V(G)$ has a separable orbit then μ is absolutely continuous with respect to m. In the case that $|\mu|(G) < \infty$ an immediate consequence of this result is the equivalence of the absolute continuity of μ and the separability of the orbit of μ.

As usual $M(G)$ will denote the space of $\mu \in V(G)$ for which $|\mu|(G) < \infty$.

2. **Lemmas.**

LEMMA 1. *Let G be a* LC *group and $\mu \in V(G)$. For each Borel set $E \subset G$ the complex-valued function $T_s\mu(E)$ is Borel measurable.*

PROOF. Let χ_E denote the characteristic function of the set E. Then $\chi_E(ts^{-1})$ is a Borel measurable function on $G \times G$ and hence $\int_G \chi_E(ts^{-1}) d\mu(t) = T_s\mu(E)$ is a Borel measurable function on G.

LEMMA 2. *Let G be a second countable* LC *group, and suppose $\mathfrak{R}$ is the ring of sets generated by some countable basis for G consisting of compact neighborhoods. If $\mu \in V(G)$ and $K \subset G$ is any compact neighborhood then $|\mu|(K) = \sup \sum_i |\mu(L_i)|$ where the supremum is taken over all finite collections of pairwise disjoint sets $L_i \in \mathfrak{R}$ whose union is contained in K.*

PROOF. It is evident that $\sup \sum_i |\mu(L_i)| \leqq |\mu|(K)$.

Conversely, suppose $\epsilon > 0$ and let g be a bounded Borel measurable function such that $|g| = 1$ and $d\mu = g d|\mu|$ [1, p. 171]. First, using the regularity of μ, we choose a compact set $C \subset K$ such that $|\mu|(K \sim C) < \epsilon/3$. Then since K is open we may apply Lusin's theorem [3, p. 53]

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to find a continuous complex-valued function f with compact support F such that (i) $C\subset F\subset K$, (ii) $\sup|f(t)|\leqq\sup|\chi_C(t)/g(t)|=1$, and (iii) $|\mu|(\{t|f(t)\neq\chi_C(t)/g(t)\})<\epsilon/6$. Finally, employing the openness of K, the regularity of μ and a standard construction for the approximation of continuous functions, we construct a function of the form $h=\sum_{i=1}^n c_i\chi_{L_i}$, where $|c_i|\leqq 1$, $L_i\in\mathfrak{R}$, $i=1, 2, \cdots, n$, $L_i\cap L_j=\varnothing$, $i\neq j$, $\bigcup_{i=1}^n L_i\subset K$, and such that $\int_G|f-h|\,d|\mu|<\epsilon/3$.

Combining these inequalities we conclude that,

$$\left|\int_G \chi_K d|\mu| - \int_G hgd|\mu|\right| \leqq \int_G |\chi_K/g - h|\,d|\mu|$$

$$\leqq \int_G |\chi_K/g - \chi_C/g|\,d|\mu| + \int_G |\chi_C/g - f|\,d|\mu|$$

$$+\int_G |f-h|\,d|\mu| < \epsilon/3 + 2\epsilon/6 + \epsilon/3 = \epsilon;$$

and hence

$$|\mu|(K) - \epsilon < \left|\int_G hgd|\mu|\right| = \left|\sum_{i=1}^n c_i\mu(L_i)\right| \leqq \sum_{i=1}^n |\mu(L_i)|\,.$$

Therefore $|\mu|(K)=\sup\sum_i|\mu(L_i)|$.

The third lemma follows immediately from the previous two lemmas.

LEMMA 3. *Let G be a second countable* LC *group and $\mu\in V(G)$. For each compact neighborhood $K\subset G$ the real-valued function $|T_s\mu-\mu|(K)$ is Borel measurable.*

3. **Theorems.** To establish the theorem mentioned in the introduction we shall need, besides Lemma 3, one further result which we state without proof. The proof given in [2, p. 230] is for abelian groups and finite measures but it is apparent that with some minor modifications of the argument the theorem given here is also valid.

THEOREM 1. *Let G be a* LC *group and $\mu\in V(G)$. Then the following are equivalent:*

(i) *μ is absolutely continuous with respect to m.*

(ii) *For each compact $K\subset G$ the function $T_s|\mu|(K)$ is continuous.*

The main result of the paper is the following theorem.

THEOREM 2. *Let G be a second countable* LC *group and $\mu\in V(G)$. If μ has a separable orbit then μ is absolutely continuous with respect to m.*

PROOF. We note first that the function $\|T_s\mu-\mu\|$ is a Borel measurable function on G. Indeed, since G is σ-compact there exists a sequence of compact neighborhoods K_i such that $K_i\subset K_{i+1}$ and $\bigcup_{i=1}^{\infty}K_i=G$. The measurability of $\|T_s\mu-\mu\|$ is then a consequence of Lemma 3 and the fact that for each $s\in G$ $\|T_s\mu-\mu\|=|T_s\mu-\mu|(G)=\lim_n|T_s\mu-\mu|(\bigcup_{i=1}^{n}K_i)$.

Let $\epsilon>0$ and set $A=\{s\mid\|T_s\mu-\mu\|\leqq\epsilon/2\}$. It is evident that A is nonempty, symmetric and, as previously remarked, measurable. Moreover A has positive Haar measure. For $s\in G$ choose $c\in C$ such that $\|T_s\mu-T_c\mu\|\leqq\epsilon/2$. A c satisfying this property exists since the orbit of μ is separable. However, $\|T_{c^{-1}s}\mu-\mu\|=\|T_{c^{-1}s}\mu-T_{c^{-1}c}\mu\|=\|T_s\mu-T_c\mu\|\leqq\epsilon/2$, and so $c^{-1}s\in A$. Thus $G=\bigcup_{c\in C}cA$, and it follows immediately that $m(A)>0$.

Combining these facts we conclude via a well-known theorem [1, p. 296] that AA contains an open neighborhood of the identity in G. Furthermore an application of the triangle inequality reveals that $AA\subset\{s\mid\|T_s\mu-\mu\|\leqq\epsilon\}$, and hence $\|T_s\mu-\mu\|$ is continuous at the identity.

Therefore for each compact $K\subset G$ the function $|T_s\mu-\mu|(K)$, and consequently also the function $T_s|\mu|(K)$, is continuous at the identity and thus at all points of G.

Finally, an application of Theorem 1 allows us to conclude that $|\mu|$, and so μ, is absolutely continuous with respect to m.

For $\mu\in M(G)$ the converse of Theorem 2 can be easily proved.

THEOREM 3. *Let G be a second countable* LC *group and $\mu\in M(G)$. Then the following are equivalent:*

(i) *μ is absolutely continuous with respect to m.*

(ii) *μ has a separable orbit.*

PROOF. The content of Theorem 2 is that (ii) implies (i).

Suppose $d\mu=f\,dm$ where f is some Borel measurable function such that $\int_G|f|\,dm<\infty$. Then the separability of the orbit of μ follows immediately from the continuity of translation of absolutely integrable functions [1, p. 285] and the second countability of G.

REMARKS. (a) If the group G is not second countable then there may exist $\mu\in M(G)$ which is absolutely continuous with respect to m but which does not have a separable orbit. An obvious example is the group $G=R_d$, the additive group of the real numbers with the discrete topology, and $\mu=\delta$, the unit mass concentrated at zero.

(b) When G is second countable and $\mu\in V(G)\sim M(G)$ then it is possible for μ to be absolutely continuous but not have a separable orbit. If $G=R$, the additive group of the real line with the usual topol-

ogy, then $d\mu(t)=e^t dt$ is such a measure. Thus the converse to Theorem 2 is not valid.

(c) On the other hand there do exist absolutely continuous measures $\mu \in V(G) \sim M(G)$, other than complex multiples of Haar measure, which have separable orbits. If we again let $G=R$ then $d\mu(t)=f(t)dt$ where $f(t)=1$, $|t|<1$, and $f(t)=1/|t|$, $|t| \geqq 1$, is one instance of such a measure.

(d) Because of the form in which Theorem 1 is stated here and in [2] it may be of some interest to note an equivalent formulation of the notion of separability for σ-compact groups, namely: there exists a countable set $C \subset G$ such that for each $s \in G$ and $\epsilon > 0$ there exists some $c \in C$ such that $|T_s\mu - T_c\mu|(K) \leqq \epsilon$ for all compact sets $K \subset G$.

References

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