A CHARACTERIZATION OF ANALYTICITY. II

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1. Introduction. We continue the study of families of analytic functions characterized by Lipschitz type conditions initiated in Analyticity. I [3]. A new property, $R$, is introduced and “smoothing” operators are developed, enabling the abstract theory to give as by-products many results of the concrete theory, including a simplified proof of the analyticity of the elements of mean value families [2], and the solution of Neumann problem for the sphere in potential theory. Employing the smoothing operators another proof is obtained for the existence of derivatives in the finite dimensional case.

A family of functions $F$ on subsets of a linear space $B$ is called an $R$ family if $F$ is invariant under transformations of $B$ of the form $x \rightarrow rx, \ r > 0$.

2. Notation and definitions. Let $B$ and $C$ be Banach spaces, and $F$ a family of continuous functions on open subsets of $B$ into $C$. Let $\mathbb{R}$ denote the reals and $\mathbb{N}$ the positive integers. $F$ is called a $T$ family if for $f, g \in F, r \in \mathbb{R}, x \in B$, and $S$ an open set in $B$, $F$ contains $rf$, the function $f+g$ defined on $\text{dom } f \cap \text{dom } g$, the restriction $f|_S$ of $f$ to $S$, and the translate $f_x$, where $f_x(y) = f(y-x)$ for $y \in \{x+t; \ t \in \text{dom } f\}$.

$F$ is called an $R$ family if for $f \in F$, $r > 0$, $F$ contains the function $g$ such that $g(x) = f(rx)$ for $x \in \{r/t; \ t \in \text{dom } f\}$.

$F$ is called an $L$ family if $F$ is a $T$ family and if for all $\delta > 0$, there exists $N(\delta) > 0$, such that $f \in F$, $M > 0$, $x \in B$, $U_x(\delta) = \{y \in B; \ |y-x| < \delta\} \subseteq \text{dom } f$, $|f(y)| \leq M$ for $y \in U_x(\delta)$, implies

\begin{equation}
|f(y) - f(x)| \leq N(\delta)M|y - x|.
\end{equation}

If in (1) “$N(\delta)$” is replaced by “$N/\delta$” $F$ is called an $L_N$ family.

$F$ is said to be closed if for all sequences $f_1, f_2, \ldots$ in $F$ with a common domain $S$, which converge uniformly on $S$ to a limit function $f_0, f_0 \in F$. For $\delta > 0$, set $U(\delta) = U_0(\delta)$ and $U = U(1)$.

3. Property $R$. Property $R$ is possessed by $L_N$ families formed from complex analytic and harmonic functions. There are however important elementary examples which do not possess property $R$; in particular, the family of solutions of the equation “$\Delta f = cf$,” $c$ fixed, $c > 0$, or $c < 0$.

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Let $B$ and $C$ be Banach spaces, $N, M > 0$, $F$ an $L_N$ family from $B$ to $C$, and $f \in F$, such that $\text{dom } f = B$, and $\|f(x)\| \leq M$ for all $x \in B$. Then for $x \in B$, $\|f(x) - f(0)\| \leq NM\|x\| \delta^{-1} \rightarrow 0$ as $\delta \rightarrow \infty$, $\delta > 0$, and hence $\|f(x) - f(0)\| = 0$, and $f$ is constant. Thus $F$ satisfies Liouville’s theorem. Since clearly all $RL$ families are $L_N$ families, $N > 0$, all $RL$ families must satisfy Liouville’s theorem. Not all $L$ families satisfy Liouville’s theorem, however. A trivial example is the $L$ family from $R$ to $R$ generated by the sine or cosine function.

We now proceed to the study of functions contained in an $RL$ family formed by differentiation and integration of elements of the family.

**Theorem 1.** Let $B$ and $C$ be Banach spaces, $n \in \omega$, $F$ a closed $TR$ family from $B$ to $C$, $\rho > 0$, and $f \in F$, such that for $i = 1, \cdots, n + 1$, $f^{(i)}$ exists and is continuous. For $x \in U(\rho)$, $i = 1, \cdots, n$, set $h_0(x) = f(0)$, and $h_i(x) = f^{(i)}_0(x, \cdots, x)$. Then $h_i \in F$ for $i = 0, 1, \cdots, n$. Moreover for some $\delta > 0$, $F$ contains the function $\theta$, such that $\theta(x) = f'_x(x)$ for $x \in U(\delta)$.

We observe from [3] that all $RL$ families satisfy the hypothesis of Theorem 1. We will need the case of $TR$ families in the study of mean value families in §5.

**Proof.** There exists $\delta > 0$ such that $U(\delta) \subseteq \text{dom } f$. For $n \in \omega$, $x \in U(n\delta)$, set $g_n(x) = f(x/n)$. Then for $p > \rho/\delta$, $p \in \omega$, the sequence $g_p, g_{p+1}, \cdots$ converges uniformly on $U(\rho)$ to $h_0$.

There exists $\delta > 0$, $M > 0$, such that $U(3\delta) \subseteq \text{dom } f$ and $\|f''_x\| \leq M$ for $x \in U(3\delta)$. Let $x \in U(3\delta)$ and for $t \in U(3\delta)$, set $g(t) = f(t) - f'_x(t - x)$. Then $g(x) = f(x)$ and for $t \in U(3\delta)$,

$$\|g_t\| = \|f_t - f'_x\| \leq \|t - x\| \sup\{\|f''_r\| ; r \in [t, x]\} \leq \|t - x\| M$$

and

$$\|f(t) - f(x) - f'_x(t - x)\| = \|g(t) - g(x)\| \leq \|t - x\| \sup\{\|g'_r\| ; r \in [t, x]\} \leq \|t - x\| \|t - x\| M = M\|t - x\|^2.$$

Let $|a| \leq 2$, and for $x \in U(\delta)$, $n \in \omega$, set $g_n(x) = n[f(ax + x/n) - f(ax)]$. Since $F$ is a $TR$ family, $g_n \in F$ for $n \in \omega$. Then for $x \in U(\delta)$, $n \in \omega$, we have $ax + x/n, ax \in U(3\delta)$, and...
Thus the sequence \( g_1, g_2, \cdots \) of \( F \) converges uniformly on \( U(\delta) \) to a limit function \( w_a \), where \( w_a(x) = f'_a(x) \) for \( x \in U(\delta) \). Since \( F \) is closed, \( w_a \in F \). Let \( r > \rho/\delta \). Then for \( x \in U(\rho) \), \( h_1(x) = f'_0(x) = rf'_0(x/r) = rw_0(x/r) \), and \( h_1 \in F \). Let \( \theta = w_1 \).

For \( \frac{a}{n} \leq 2, x \in U(\delta/3), n \in \omega \), replacing \( g_n(x) \) by \( n [f'_{a+1/n}(x) - f'_a(x)] \) \( = n [w_{a+1/n}(x) - w_a(x)] \), and setting \( w_{1,a}(x) = f'_a(x, x) \), we obtain, when \( a = 0, h_2 \in F \). Continuing this process the theorem follows.

**Theorem 2.** Let \( B \) and \( C \) be Banach spaces, \( F \) a closed RL family from \( B \) to \( C \), and \( f \) a uniformly continuous function on \( U \) to \( C \), such that \( f(0) = 0 \), and \( f \mid U \in F \). Then there exists a uniformly continuous function \( g \) on \( U \) to \( C \), such that \( g \mid U \in F \), \( g'_a(x) = f(x) \) for all \( x \in U \), and \( \lim_{r \to 0}[g(x) - g(rx)]/(1 - r) = f(x) \) for all \( x \in U - U \).

**Proof.** For \( x \in U \), \( 0 < r \leq 1 \), set \( r_s(x) = f(rx)/r \), and set \( h_0(x) = f'_0(x) \).

Let \( \varepsilon > 0 \). Then there exists \( 0 < \rho < 1 \), such that for \( x \in U(\rho) \), \( \|f(x) - f(0) - f'_0(x)\| \leq \varepsilon \|x\|/2 \). Then for \( 0 < r, s \leq \rho, x \in U \), we have \( rx, sx \in U \), and

\[
\|h_s(x) - h_r(x)\| = \|f(sx)/s - f(rx)/r\|
\leq s^{-1}\|f(sx) - f'_0(sx)\| + r^{-1}\|f(rx) - f'_0(rx)\|
\leq s^{-1}[\varepsilon \|sx\|/2] + r^{-1}[\varepsilon \|rx\|/2]
= \varepsilon/2 + \varepsilon/2 = \varepsilon.
\]

Since \( f \) is uniformly continuous, there exists \( 0 < \delta < \rho \), such that for \( \rho \leq r \leq s \leq 1, \ |s - r| \leq \delta, \) and \( x \in U \), we have \( \|h_s(x) - h_r(x)\| \leq \varepsilon \).

For \( x \in U \), set \( g(x) = \int_0^1 h_r(x) \, dr \), (1). For \( x \in U \), \( n > 1/\delta, n \in \omega \),

\[
\|g(x) - \sum_{0}^{n-1} (1/n) h_{i/n}(x)\| \leq \sum_{0}^{n-1} \left| \int_{i/n}^{(i+1)/n} h_r(x) \, dr - (1/n) h_{i/n}(x) \right|
\leq \sum_{0}^{n-1} \left| \int_{i/n}^{(i+1)/n} [h_r(x) - h_{i/n}(x)] \, dr \right|
\leq \sum_{0}^{n-1} \varepsilon/n = n[\varepsilon/n] = \varepsilon,
\]

and thus \( g \mid U \) is the limit on \( U \) of the uniformly convergent sequence \( \{ \sum_{0}^{n-1} (1/n) h_{i/n} \} \cap U; n \in \omega \} \) in \( F \). Since \( F \) is closed, \( g \mid U \in F \).

For \( x \in U, x \neq 0 \), and \( n \in \omega, 1/n < 1 - \|x\| \), substituting \( s = r \|x\| \) in
(1), we have \( g(x) = \int_0^{||x||} f(sx/||x||) s^{-1} ds \) and

\[
g(x + x/n) = \int_0^{||x|| + (1+1/n)} f(sx(1 + 1/n)/||x + x/n||) s^{-1} ds
\]

and thus

\[
g'(x) = \lim_{n \to \infty} n[g(x + x/n) - g(x)]
\]

\[
= ||x|| \lim_{n \to \infty} ||x/n||^{-1} \int_0^{||x|| + (1+1/n)} f(sx/||x||) s^{-1} ds
\]

\[
= ||x|| \left[ f(||x|| x/||x||) ||x||^{-1} \right] = f(x).
\]

**Remark.** Theorem 2 may be used to solve the Neumann problem in potential theory for the sphere once the Dirichlet problem is solved for the sphere. Let \( E \) be a Euclidean space, and let \( \phi \) be a map (continuous function) of \( H = \overline{U} \cap U \) into \( \mathbb{R} \), such that \( \int_U \phi d\mu = 0 \), where \( \mu \) is normalized surface measure on \( H \). Let \( f \) be a map on \( \overline{U} \), such that \( f \big| U \) is harmonic and \( f \big| H = \phi \). Then \( f(0) = \int_U f d\mu = \int_H \phi d\mu = 0 \). Since the family of harmonic functions on open subsets of \( E \) into \( \mathbb{R} \) is an \( RL \) family, from Theorem 2, there exists a map \( g \) on \( \overline{U} \), such that \( g \big| U \) is harmonic, and

\[
\lim_{r \to 1} [g(rx) - g(rx)]/(1 - r) = f(x) = \phi(x) \quad \text{for } x \in H.
\]

**4. Smoothing operators.** The operators introduced in this section will enable us to approximate elements of a closed \( T \) family \( F \) by elements of \( F \) which are as smooth, i.e., differentiable, as desired. They are similar to averaging operators used in potential theory [6] with the difference that here averages are taken over cubes rather than spheres.

**Definition 1.** Throughout the remainder of this paper \( E \) shall denote a fixed Euclidean space, and \( e_1, \ldots, e_p, p \in \omega \), a fixed orthonormal basis of \( E \). Set \( Q = \{ x \in E; -1/2 \leq [x, e_i] \leq 1/2, i = 1, \ldots, p \} \) and for \( i = 1, \ldots, p \), set \( Q_i = \{ x \in Q; [x, e_i] = 0 \} \).

Let \( f \) be a map of an open set \( S \) in \( E \) into a Banach space \( B \). Then for \( x \in E, a > 0 \), such that \( x + aQ = \{ x + ay; y \in Q \} \subseteq S \), set \( L(f, a)(x) = a^{-p} \int_{x+aQ} f(t) dm(t) \), where \( m \) is Lebesgue measure on \( E \).

**Theorem 3.** Let \( B \) be a Banach space, \( F \) a closed \( T \) family from \( E \) to
B, \(a>0\), and \(f \in F\). Set \(g = L(f, a)\) and let \(H\) be a compact subset of the domain of \(g\) with interior \(S\). Then \(g \mid S\) lies in \(F\) and \(g\) is continuously differentiable. If \(f\) is continuously differentiable, \(g' = [L(f, a)]' = L(f', a)\). Thus \(L(g, a)''\) and \(L[L(g, a), a]''\) exist and are continuous.

Moreover, for \(0<s<a\), \(f_s = L(f, s)\) converges uniformly on \(H\) to \(f \mid H\) as \(s \to 0\).

**Proof.** Let \(\varepsilon > 0\) and set \(M = H + aQ\). Then \(M \subseteq \text{dom}\, f\), and there exists \(\delta > 0\), such that \(x, y \in M\), \(\|y - x\| < \delta\), implies \(\|f(y) - f(x)\| \leq \varepsilon\). Let \(0<r<\delta/p^{1/2}\), and \(t_1, \ldots, t_n \in aQ\), \(n \in \omega\), such that \(\{t_i + rQ; i = 1, \ldots, n\}\) is a subdivision of \(aQ\). Then for \(x \in H, i = 1, \ldots, n\), \(t_i \in t_i + rQ\), we have \(x + t_i \in x + t_i + rQ \subseteq x + aQ \subseteq H + aQ = M\), \(\|(x + t_i) - (x + t_i)\| \leq r \leq p^{1/2}\delta < \delta\), and \(\|f(x + t_i) - f(x + t_i)\| \leq \varepsilon\). Thus for \(x \in H\),

\[
\left\| \sum_{1}^{n} \frac{r}{a} f_{-t_i}(x) - g(x) \right\| \leq a^{-p} \left\| \sum_{1}^{n} f(x + t_i) a^{-p} - \int_{t_i + rQ} f(x + t) dm(t) \right\|
\]

\[
\leq a^{-p} \sum_{1}^{n} \left\| \int_{t_i + rQ} [f(x + t_i) - f(x + t)] dm(t) \right\|
\]

\[
\leq a^{-p} \sum_{1}^{n} \epsilon r a^{-p} = a^{-p} [\epsilon a^p] = \epsilon.
\]

Thus \(g \mid S\) is the uniform limit on \(S\) of elements of \(F\) of the form \(\sum_{1}^{n} \frac{r}{a} f_{-t_i}\). Since \(F\) is closed, \(g \mid S \in F\).

Similarly for \(s > 0\), \(s > a\), \(\delta\), and \(x \in H\),

\[
\|f(x) - f_s(x)\| = s^{-p} \left\| \int_{aQ} [f(x) - f(x + t)] dm(t) \right\| \leq s^{-p} [\epsilon s^p] = \epsilon,
\]

and thus \(f_s\) converges uniformly on \(H\) to \(f\) as \(s \to 0\), \(s < a\).

Let \(i = 1, \ldots, p\), and set \(\rho = a/2\). Then \(M_i = H + [-\rho, \rho] e_i \subseteq M_i\). For \(x \in M_i\), set \(g_i(x) = a^{-p} \int_{aQ} f(x + t) dm(t)\). Then for \(x, y \in M_i\), \(\|y - x\| < \delta\),

\[
\|g_i(y) - g_i(x)\| = a^{-p} \left\| \int_{aQ} [f(y + t) - f(x + t)] dm(t) \right\|
\]

\[
\leq a^{-p} [\epsilon a^p] = \epsilon/a.
\]

For \(x \in H\), \(t \in E\), set \(A_x(t) = \sum_{1}^{p} \left[ t_i, e_i \right] [g_i(x + \rho e_i) - g_i(x - \rho e_i)]\). From (1), the function \(A : x \to A_x (x \in H)\) is continuous. Then for \(x \in S\), \(t \in E\), such that \(U_x(|t|) \subseteq S\), setting \(s_i = [t, e_i]\) for \(i = 1, \ldots, p\), and setting \(t_1 = x\), and \(t_i = x + \sum_{i-1}^{1} s_j e_j\) for \(i = 2, \ldots, p + 1\), we have \(t_i \in S\) for \(i = 1, \ldots, p + 1\), and \(\|s_j e_i\| < \delta\) for \(i = 1, \ldots, p\) and
\[ \|g(x + t) - g(x) - A_{x}(t)\| = \sum_{i=1}^{p} \|g(t_{i+1}) - g(t_{i}) - A_{x}(t_{i+1} - t_{i})\| \]
\[ \leq \sum_{i=1}^{p} \|g(t_{i} + s_{i}e_{i}) - g(t_{i}) - A_{x}(s_{i}e_{i})\| \]
\[ \leq \sum_{i=1}^{p} \left| \int_{-p+s_{i}}^{p+s_{i}} g_{i}(t_{i} + se_{i})ds - \int_{-p}^{p} g_{i}(t_{i} + se_{i})ds \right| \]
\[ = \sum_{i=1}^{p} \left| \int_{-p}^{p} \left[ g_{i}(t_{i} + se_{i}) - g_{i}(x + pe_{i}) \right]ds \right| \]
\[ \leq \sum_{i=1}^{p} \left| s_{i} \right| \left( \epsilon/a \right) + \left| s_{i} \right| \left( \epsilon/a \right) \leq \sum_{i=1}^{p} 2 ||\epsilon/a|| = 2pa^{-1}||\epsilon||, \]

and thus \( g'_{x} \) exists and \( g'_{x} = A_{x} \), and \( g \) is continuously differentiable.

Assume that \( f \) is continuously differentiable and let \( \epsilon > 0 \). Then there exists \( \rho > 0 \), such that \( x, y \in M, \|y - x\| < \rho \), implies \( \|f(y) - f(x) - f'_{x}(y - x)\| \leq \epsilon \|y - x\| \). Then for \( x \in S, t \in U(\rho) \), such that \( x + t \in S \),
\[ \|g(x + t) - g(x) - L(f', a)_{x}(t)\| \]
\[ = \left| a^{-p} \int_{x+a^{p}} [f(t + s) - f(s) - f'_{x}(t)]dm(t) \right| \]
\[ \leq a^{-p} \|\epsilon\|a^{p} = \epsilon \|t\|, \]

and \( g'_{x} = L(f', a)_{x} \).

Remark. Let \( B \) be a Banach space and \( F \) an \( L \) family from \( E \) to \( B \). Then Theorem 3 may be used to give another proof [3] of the existence of derivatives of elements of \( F \). Let \( f \in F \). For \( n \in \omega \), set \( f_{n} = L(f, 1/2^{n}) \). Let \( x \in \text{dom } f \), \( \delta > 0 \), such that \( H = U_{x}(\delta) \subseteq \text{dom } f \). Then there exists \( g \in \omega \), such that \( n \geq q, n \in \omega \), implies \( H \subseteq \text{dom } f_{q} \). From Theorem 3, for \( n \in \omega \), \( f_{n} \) lies in \( F \) and is differentiable, and the sequence \( f_{q}, f_{q+1}, \cdots \) converges uniformly on \( H \) to \( f \). From Theorem
3.4 of [3], for $t \in U_x(\delta), f_t'$ exists, and the sequence $(f_q)_t', (f_{q+1})_t', \ldots$ converges to $f_t'$.

An extension to the infinite dimensional case, when $E$ is replaced by an arbitrary Banach space, is possible but tedious.

A useful provider of examples is the following consequence of Theorem 3.

**Theorem 4.** Let $B$ be a Banach space, and $F$ a closed $T$ family from $E$ to $B$, such that for $x \in E, \delta > 0, \{f \mid U_x(\delta); U_x(\delta) \subseteq \text{dom } f, f \in F\}$ is finite dimensional. Then $F$ is an $L$ family.

**Proof.** Let $\delta > 0$, set $H = \overline{U}(\delta)$, and set $G = \{L(f, a) \mid H; a > 0, f \in F, H \subseteq \text{dom } L(f, a)\}$. Then $G$ is a finite dimensional linear space, and from Theorem 3, the elements of $G$ are continuously differentiable on $H$. Let $g_1, \ldots, g_n, n \in \omega$, be a basis of $G$ and set $N = \sup \{\|g_i\|_t' ; t \in H\}$. Then $N < \infty$, and for $i = 1, \ldots, n, y \in H$,

$$\|g_i(y) - g_i(0)\| \leq \|y\| \sup\{\|g_i\|_t' ; t \in [0, y]\} \leq \|y\|N.$$  

For $f \in G$, there exists a unique sequence of numbers $L_1(f), \ldots, L_n(f)$, such that $f = \sum_1^n L_i(f)g_i$. Since $G$ is finite dimensional and $L_1, \ldots, L_n$ are linear, $L = \sup \{\|L_i\| ; i = 1, \ldots, n\} < \infty$. Thus for $f \in G, M > 0$, such that $\|f(y)\| \leq M$ for $y \in H$,

$$\|f(y) - f(0)\| = \left\| \sum_1^n L_i(f) \left[ g_i(y) - g_i(0) \right] \right\|$$

$$\leq \sum_1^n \|L_i(f)\| \|g_i(y) - g_i(0)\|$$

$$\leq \sum_1^n [LM] \cdot [N\|y\|] = N(\delta)M\|y\|,$$

where $N(\delta) = nNL$.

Let $f$ be an element of $F$ such that $H \subseteq \text{dom } f$. Then from Theorem 3, $f|_H$ is the uniform limit on $H$ of elements of $G$, and hence $f$ satisfies (1), and $F$ is an $L$ family.

**Remark.** The argument for Theorem 4 is somewhat simplified if for $E$ and $B$ we take $R$, and for $G$ we take the family of antiderivatives (indefinite integrals) of elements of $F$. In this case [1] all elements of $F$ are linear combinations of expressions of the form $x^p e^{ax}$, where $p = 0, 1, \ldots$, and $a$ is arbitrary.

5. **Volume mean families.**

**Theorem 5.** Let $\mu$ be a nonnegative Borel measure on $E$, with compact
support $K$, such that $K$ is contained in no proper subspace of $E$, and
$\mu(K) = 1$. Let $F$ be the family of all maps $f$ of open sets in $E$ into $R$, such
that for $x \in E$, $\delta > 0$, if $x + \delta K \subseteq \text{dom} f$,
\[
f(x) = \int_K f(x + \delta t)d\mu(t).
\]
Then the elements of $F$ satisfy an elliptic partial differential equation
and are analytic.

A proof of this result involving Fourier transforms and the notion
of weak solutions of Laplace's equation is given by Friedman and
Littman [2]. A considerably simpler argument is given here using the
machinery developed in this paper.

PROOF. Trivially $F$ is a closed $TR$ family. Set $G = \{L[L(L(f, a), a), a];
a > 0, f \in F\}$. From Theorem 3, $G$ is a $TR$ family, $g^{(3)}$ exists and is
continuous for $g \in G$, and $f \in F$, $S \subseteq E$, $S$ compact, $\bar{S} \subseteq \text{dom} f$, implies
$f|S$ lies in the closure $G_0$ of $G$.

Let $H$ be the space of all symmetric bilinear functionals on $E \times E$
into $R$. For $\theta \in H$, set $P(\theta) = \int_K \theta(t, t)d\mu(t)$. For $x, y, r, s \in E$, set
$\theta[x, y](r, s) = [x, r][y, s]$, and set $I'(x, y) = [x, y]' = P(\theta[x, y])$. Since
$\mu$ is not supported on any proper subspace of $E$, $[x, x]' > 0$ for all
$x \in E$, $x \neq 0$. Thus $I'$ is an inner product on $E$. Let $E'$ denote the new
Euclidean space $\{E, I'\}$ determined by $I'$.

Let $T$ be a unitary transformation (rotation) of $E'$ into itself. Then
for $x, y \in E'$, $[Tx, Ty]' = [x, y]'$, and $P(\theta[Tx, Ty]) = [Tx, Ty]' = [x, y]' = P(\theta[x, y])$. Now the collection $\{\theta[x, y]; x, y \in E\}$ generates
$H$. Thus with respect to $E'$, $P$ is a rotation invariant operator operating
on $H^*$, and hence letting $e_1', \cdots, e_p'$ be an arbitrary orthonormal
basis of $E'$, there exists $c \neq 0$ such that $P(\theta) = c \sum \theta(e_i', e_i')$ for all
$\theta \in H$.

Let $f \in G$, $x \in \text{dom} f$, and $\rho > \sup \{\|t\|; t \in K\}$, and set $h(t) = f_x''(t, t)$
for $t \in U(\rho)$. From Theorem 1, $h \in F$, and
\[
\sum c f_x''(e_i', e_i') = P(f_x'') = \int_K f_x''(t, t)d\mu(t) = \int_K h(t)d\mu(t) = h(0) = f_x''(0, 0) = 0.
\]
Thus the elements of $G$ are harmonic functions with respect to $E'$.
From [4], [5], the elements of $G_0$ and hence of $F$ are harmonic func-
tions with respect to $E'$, and hence analytic.
Remarks. Theorems 4 and 5 can also be handled from the standpoint of distribution theory [7]. We consider the family of functions $F$ in question as a family of distributions and take the closure $F_0$ of $F$ in the weak or distribution sense. In Theorem 5, employing property $R$, we observe that $F_0$ contains the functions $h_x(t) = f'_x(t, t)$ which exist at least weakly. We then show that the elements of $F$ satisfy an elliptic partial differential equation at least weakly, and hence strongly.

In Theorem 4 [1], restricting attention to functions with real range, we have that the weak partial derivatives of the elements of $F$ lie in $F_0$. Since $F$ is finite dimensional, $F_0 = F$, and hence the elements of $F$ are strongly differentiable.

The smoothing operators introduced in §4 can be considered as analogues of convolutions of distributions with suitable approximate identity functions.

References


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