AN OSCILLATION CRITERION FOR SECOND-ORDER LINEAR DIFFERENTIAL EQUATIONS

W. J. COLES


(1) \[ y'' + p(x)y = 0 \quad p(x) \text{ continuous on } [0, \infty) \]

is

Theorem (i). Equation (1) is oscillatory on \([0, \infty)\) if

(2) \[ \int_{0}^{\infty} p = + \infty. \]

The case

(3) \[ \liminf_{x \to \infty} \int_{x}^{\infty} p < + \infty \]

remains of interest and can produce either oscillatory or nonoscillatory behavior.

Let

(4) \[ P(x) = \frac{1}{x} \int_{0}^{x} \int_{0}^{t} p(s)dsdt. \]

Hartman [1] has proved that nonoscillation of (1) implies that either \(P(x)\) tends to a finite limit or else that \(\lim_{x \to \infty} P(x) = -\infty\), so that one has:

Theorem (ii). \(-\infty < \liminf_{x \to \infty} P(x) < \limsup_{x \to \infty} P(x)\) implies oscillation.

Theorem (iii). \(\lim_{x \to \infty} P(x) = +\infty\) implies oscillation.

Since (2) implies the hypothesis of (iii), Theorem (iii) implies Theorem (i).

The above theorems do not apply if \(P(x)\) tends to a finite limit or if \(\liminf_{x \to \infty} P(x) = -\infty\); e.g., they give no information about such coefficients as \(p(x) = (x \cos x - \sin x)/x^2\) or \(p(x) = x^2 \sin x\). The purpose of this note is to derive oscillation criteria for certain classes of such coefficients.

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2. **Weighted averages.** The idea is that additional information about oscillation of (1) may be gained by considering weighted averages of $f^zp$. Let $f$ be a nonnegative, locally integrable function such that $\int_{a}^{x} f \neq 0$; then there is an $a > 0$ such that

\[(5) \quad A(x) = A(f; p)(x) = \int_{0}^{x} f(t) \int_{0}^{t} p(s)ds dt \bigg/ \int_{0}^{x} f(t)dt\]

exists on $[a, \infty)$.

**Theorem 1.** If there exists a nonnegative, locally integrable function $f$ satisfying

\[(6) \quad \int_{a}^{\infty} \left\{ f(t) \left( \int_{0}^{t} f(s)ds \right)^{k} \bigg/ \int_{0}^{t} f^{2}(s)ds \right\} dt = + \infty\]

for some $k$, $0 \leq k < 1$, and for $a > 0$

and

\[(7) \quad \lim_{x \to \infty} A(x) = + \infty,\]

then (1) is oscillatory.

**Proof.** The proof uses ideas of Hartman. We give a proof for $f$ continuous; the proof is easily modified for $f$ locally integrable. Also, if convenient, we will change the lower limits of the integrals in (5) and (6), since the asymptotic behavior as $x \to \infty$ is not changed thereby.

Suppose that (1) is nonoscillatory; then, for large enough $a$, a solution of the Riccati equation

\[(8) \quad z' + z^2 + p(x) = 0\]

exists on $[a, \infty)$. Integration, multiplication by $f$, and integration give

\[(9) \quad \int_{a}^{x} f(t)z(t)dt + \int_{a}^{x} f(t) \int_{a}^{t} z^2(s)ds dt = (z(a) - A(x)) \int_{a}^{x} f(t)dt.\]

By hypothesis, the right-hand side tends to $-\infty$; hence, for large enough $x$,

\[\int_{a}^{x} f(t)z(t)dt + \int_{a}^{x} f(t) \int_{a}^{t} z^2(s)ds dt < 0\]

so that
\[
\left( \int_a^z f(t) \int_t^z s^2(s) ds \, dt \right)^2 \leq \left( \int_a^z f(t) s(t) dt \right)^2 \leq \int_a^z f^2(t) dt \cdot \int_a^z s^2(t) dt.
\]

(10)

Let \( R(x) = \int_a^z f(t) \int_t^z s^2(s) ds \, dt \). Since, for \( x \geq b > a \), \( R(x) \geq \int_b^z f(t) dt \cdot \int_a^z s^2(t) dt \), we have from (10) that

\[
f(x) \left( \int_b^z f(t) dt \right)^k \left( \int_a^b s^2(t) dt \right)^k \int_a^z f^2(t) dt \leq R^{k-2}(x) R'(x).
\]

For \( b > a \), integration now gives

\[
\int_b^z \left\{ f(t) \left( \int_b^t f(s) ds \right)^k \right\} \int_a^t f^2(s) ds \leq \frac{1}{h} \left( \frac{1}{R^h(b)} - \frac{1}{R^h(x)} \right) < \frac{1}{h R^h(b)} \quad (k = 1 - k)
\]

contradicting (6).

Note that (6) implies that

\[
\int_a^\infty f(t) dt = + \infty,
\]

a reasonable condition for a weight function. Conversely, if \( f \) is bounded then (11) implies (6) for any \( k \) such that \( 0 \leq k < 1 \).

If (6) holds for some \( k \) on \([0, 1]\), it holds for \( k = 0 \); but one advantage in stating the theorem for \( k > 0 \) is that weight functions \( x^\alpha (\alpha > 0) \) are permitted.

The following corollary says roughly that if \( \int_0^z \phi \) is large enough on a large enough set, then (1) is oscillatory regardless of the behavior of \( \int_0^z \phi \) on the rest of the half line.

**Corollary 1.** Let \( S(x) = \{ t \mid 0 \leq t \leq x \text{ and } \int_0^t \phi > 0 \} \), and let \( m(S(x)) \) be the measure of \( S(x) \). If \( m(S(x)) \to \infty \) as \( x \to \infty \) and if

\[
\frac{1}{m(S(x))} \int_{S(x)} \int_t^z \phi(s) ds \, dt \to \infty
\]

as \( x \to \infty \), then (1) is oscillatory.

**Proof.** Take \( f(x) \) to be 1 if \( \int_0^z \phi > 0 \) and 0 otherwise, let \( k = 0 \), and apply Theorem 1.
Examples. If \( p(x) = x \sin x \), Theorems (i) and (iii) fail, but Theorem (ii) and Corollary 1 apply.

If \( p(x) = x^2 \sin x \), Theorems (i), (ii), and (iii) fail, but Corollary 1 applies.

Even if no suitable weight function exists for Theorem 1, (1) may oscillate, as the following theorem implies.

**Theorem 2.** If \( P(x) \) does not approach a finite limit as \( x \to \infty \) and if there is a nonnegative, locally integrable function \( f \) satisfying (6) and

\[
\lim_{x \to \infty} \inf A(x) > -\infty,
\]

then (1) is oscillatory.

The interesting way for \( P(x) \) to fail to have a limit is

\[
\lim_{x \to \infty} \inf P(x) = -\infty;
\]

this is the only case not covered by Theorems (ii) and (iii).

**Proof of Theorem 2.** First we remark that since \( \int_{a}^{x} f = +\infty \) if \( g(x) \) is nondecreasing in \( x \), we have:

(a) \( \int_{a}^{x} f(t)g(t)dt/\int_{a}^{x} f(t)dt \) is nondecreasing in \( x \);

(b) if \( \int_{a}^{x} f(t)g(t)dt/\int_{a}^{x} f(t)dt \) is bounded on \([a, \infty)\), so is \( g(x) \).

Suppose that (1) is nonoscillatory. Via the Riccati equation we have (by (12))

\[
\left[ \int_{a}^{x} f(t)z(t)dt + \int_{a}^{x} f(t) \int_{t}^{x} z(s)dsdt \right]/\int_{a}^{x} f(t)dt = z(a) - A(x) \leq K \quad (K \text{ constant}) \text{ on } [b, \infty), b > a.
\]

We claim that \( \int_{a}^{x} f(t) \int_{t}^{x} z^2(s)dsdt/\int_{a}^{x} f(t)dt \) is bounded on \([b, \infty)\). If not, by (a), it tends to +\infty; and so, for large \( x \),

\[
\int_{a}^{x} f(t)z(t)dt + \frac{1}{2} \int_{a}^{x} f(t) \int_{t}^{x} z^2(s)dsdt \leq \left( K - \left( \int_{a}^{x} f(t) \int_{t}^{x} z^2(s)dsdt/2 \int_{a}^{x} f(t)dt \right) \right) \int_{a}^{x} f(t)dt < 0.
\]

Now one proceeds as in the proof of Theorem 1 to contradict (6).

So, by (b), \( \int_{a}^{x} z^2 < \infty \). But Hartman [1] has shown that if (1) is nonoscillatory, then \( \int_{a}^{x} z^2 < \infty \) if and only if \( P(x) \) has a finite limit as \( x \to \infty \). This contradiction completes the proof.

**Example.** Let \( p(x) \) be such that \( \int_{a}^{x} p \) is 0 on \([2n, 2n+1] \) \((n = 0, 1, \cdots)\) but is sufficiently negative on \((2n+1, 2n+2)\)
(n = 0, 1, · · · ) to produce \( \lim \inf_{x \to \infty} P(x) = -\infty \). Let \( f(x) \) be 1 on \([2n, 2n+1]\) (\( n = 0, 1, \cdot \cdot \cdot \)) and 0 elsewhere so that \( A(x) \equiv 0 \) on \((0, \infty)\). Theorems (i), (ii), (iii), and 1 do not apply, but Theorem 2 does.

3. The general selfadjoint case. Corresponding to Theorems 1 and 2 are Theorems 1° and 2° for the equation

\[(1°) \quad (r(x)y')' + p(x)y = 0 \quad (r(x) > 0; \quad r(x) \text{ and } p(x) \text{ continuous on } [0, \infty)).\]

Equation (6) becomes

\[(6°) \quad \int_0^\infty \left\{ f(t) \left( \int_0^t f(s)ds \right)^k \right\} \int_0^t r(s)f^2(s)ds \right\} dt = + \infty.\]

The proofs parallel those of Theorems 1 and 2 so are omitted here. The result of Hartman, needed in the proof of Theorem 2, holds for (1°) with \( \int_{-\infty}^{\infty} z^2 \) replaced by \( \int_{-\infty}^{\infty} z^2/r \).

The Leighton-Wintner theorem for (1°), namely that (1°) is oscillatory if \( \int_{-\infty}^{\infty} p = \int_{-\infty}^{\infty} 1/r = + \infty \), follows from Theorem 1° on taking \( f = 1/r \) and \( k = 0 \). On the other hand, if (6°) holds for some \( f \), then \( \int_{-\infty}^{\infty} 1/r = + \infty \); and (7) implies that \( \lim \sup_{x \to \infty} \int z^p = + \infty \). Thus Theorem 1° (and Theorem 1) are interesting only in case \( \int_{-\infty}^{\infty} (1/r) = + \infty \), \( \lim \inf_{x \to \infty} \int z^p < \lim \sup_{x \to \infty} \int z^p = + \infty \).

As with Theorem 2, the interesting case of Theorem 2° is \( \lim \inf_{x \to \infty} P(x) = - \infty \).

References


University of Utah