BERNSTEIN'S THEOREM FOR BANACH SPACES

ROBERT WELLAND

This paper contains a Banach space generalization of Bernstein's Theorem. *An absolutely monotonic* ($f^{(n)} \geqq 0$, $n=1, 2, \cdots$) *real valued function of a real variable is real analytic.*

For some background on this theorem, see S. Bernstein [**1**], D. V. Widder [**5**], or R. Boas [**2**].

We first present some definitions. We denote a Banach space over the reals by E; U is an open subset of E; f is a real valued function whose domain is U and $D^k f_x$ denotes the kth derivative of f evaluated at x where $k=0, 1, 2, \cdots$ [**4**].

A function f is analytic at x in U if there exists $r>0$ such that the Taylor series for f at x

$$f(z) = \sum_{k=0}^{\infty} D^k f_x (z-x)^k / k!$$

converges to f for $\|z-x\| < r$, where $(z-x)^k = (x-z, x-z, \cdots, x-z)$ is an element of E^k [**6**]. This is not the definition given in [**6**] of analyticity but it is shown there to be equivalent.

Let C be a convex cone [**3**] in E. We say C is regular if there exists a number $M>0$ such that z in E implies there exists a p in C such that $\|p\| \leqq M\|z\|$ and $p+z \in C$. The positive cone in any topological vector lattice has this property.

Lastly, if f is a C^∞ function in U (all derivatives exist at every point of U and as functions of x in U are continuous), then we say that f is absolutely monotonic relative to C if

$$D^k f_x(p_1, p_2, \cdots, p_k) \geqq 0$$

for every x in U and $(p_1, p_2, \cdots, p_k)$ in C^k, $k=1, 2, \cdots$.

THEOREM 1. *Let E be a Banach space over the reals which has a regular positive cone C. If f is a real valued C^∞-function defined on an open subset U of E and if f is absolutely monotonic relative to C, then f is analytic.*

PROOF. Because translation is analytic and because its inverse exists and is analytic, we can assume that $0 \in U$ and have only to show the Taylor series converges to f in a neighborhood of 0.

Taylor's formula, when the domain and range are Banach spaces [**4**], is

Received by the editors March 6, 1967.

$$f(x) = f(0) + Df_0(0) + \cdots + \frac{1}{(k-1)!} D^{k-1} f_0(0)^{k-1} + R_k^f(x),$$

where

$$R_k^f(x) = \frac{1}{(k-1)!} \int_0^1 (1-t)^{k-1} D^k f_{tx}(x)^k \, dt;$$

so we must show that $R_k^f(x)$ converges uniformly to 0 in some ball $B_r = \{x : \|x\| \leqq r\}$, $r > 0$.

First we prove two lemmas.

LEMMA 1. *Suppose $P = (p_1, p_2, \cdots, p_k) \in C^k$, suppose $p \in C$ and suppose that x is an arbitrary element of E. Then*

$$g(s) = D^k f_{x+sp}(P) \tag{1}$$

is a nondecreasing nonnegative function of s for every $k = 0, 1, \cdots$.

PROOF. We have only to apply the chain rule to the right-hand side of 1 to obtain

$$g'(s) = D^{k+1} f_{x+sp}(p, P) \geqq 0$$

where $(p, P) = (p, p_1, p_2, \cdots, p_k)$.

LEMMA 2. *Let x be any arbitrary point in E and let p be an element of C such that $p + x \in C$. Then*

$$\left| D^k f_{tx}(x)^k \right| \leqq D^k f_{tz}(z)^k \tag{2}$$

for $k = 0, 1, 2, \cdots$ and $z = p + x + p$.

PROOF. We begin by observing that C is convex so that $(x+p)/2 + p/2 \in C$; because $2c \in C$ for every c in C, we have $z \in C$. Next

$$\left| D^k f_{tx}(x + p - p)^k \right| \leqq \sum_{r=0}^{k} C_r^k D^k f_{tx}((x+p)^{k-r}, p^r) = D^k f_{tx}(z)^k,$$

where the C_r^k are the binomial coefficients. Here we have used the facts that $D^k f$ is k-linear and symmetric [4]. Now applying Lemma 1 to the function

$$g(s) = D^k f_{(1-s)tx + stx}(z)^k = D^k f_{tx+s2tp}(z)^k,$$

we find that

$$D^k f_{tx}(z)^k \leqq D^k f_{tz}(z)^k.$$

This completes the proof.

We now complete the proof of the theorem.

Because f is continuous at 0, there exists constants $K>0$ and $r>0$ such that $|f(x)|\leqq K$ whenever $x\in B_{2(M+1)r}$.

Let x be an arbitrary element of B_r. Choose $p\in C$ such that $x+p\in C$ and $\|p\|\leqq M\|x\|$. By Lemma 2, we have that

$$|R_k^f(x)| \leqq R_k^f(z). \tag{3}$$

Applying Lemma 2 with $x=z$ and $p=z/2$, yields

$$2^{-k}R_k^f(z) \leqq R_k^f(2z). \tag{4}$$

Since $\|2z\|=2\|x+2p\|\leqq 2(r+2M)$, $|f(2z)|\leqq K$. Now $f(2z)=\alpha+R_k^f(2z)$, $\alpha\geqq 0$; this implies $R_k^f(2z)\leqq K$. Putting this together with 3 and 4, we see that $|R_k^f(x)|\leqq 2^{-k}K$ for every x in B_r.

If E_1 and E_2 are Banach spaces with convex cones C_1 and C_2 respectively, then we say $\bar{f}$: $U\rightarrow E_2$ is absolutely monotonic relative to (C_1, C_2) if

$$D^k\bar{f}_x(p_1, p_2, \cdots, p_k) \in C_2$$

whenever $(p_1, p_2, \cdots, p_k)\in C_1^k$.

Define the adjoint cone of C_2 by

$$C_2^* = \{y^* \in C_2^* : y^*(y) \geqq 0 \text{ for all } y \text{ in } C_2\},$$

where E_2^* is the dual of E_2.

THEOREM 2. *Let E_1 and E_2 be Banach spaces with regular cones C_1 and C_2, respectively. Let U be an open subset of E_1 and $\bar{f}$: $U\rightarrow E_2$ a C^∞-function which is absolutely monotonic relative to (C_1, C_2). If C_2^* is a regular cone, then $\bar{f}$ is an analytic function in U.*

PROOF. Let y^* be an arbitrary element in E_2^* such that $\|y^*\|\leqq 1$. Because C_2^* is a regular cone there exists a constant S independent of y^* and a point p^* in C_2^* such that $y^*+p^*\in C_2^*$ and $\|p^*\|\leqq S\|y^*\|$. Let $y^*+p^*=z^*$. Now when $v^*\in C_2^*$, $v^*\odot\bar{f}$ satisfies the hypotheses of the previous theorem so that

$$|R_k^{v^*\cdot\bar{f}}(x)| \leqq 2^{-k}K\|v^*\|$$

for all x in B_r. However, $R_k^{v^*\cdot\bar{f}}(x)=y^*R_k^{\bar{f}}(x)=(z^*-p^*)R_k^{\bar{f}}(x)$; so that

$$\|R_k^{\bar{f}}(x)\| = \sup_{\|y^*\|\leqq 1} |y^*R_k^{\bar{f}}(x)| \leqq 2^{-k}K(2S+1).$$

I do not know to what extent the regularity conditions on the cones in Theorem 2 are necessary.

References

1. S. Bernstein, *Sur les fonctions absolument monotones*, Acta. Math. **52** (1928), 56–66.

2. R. P. Boas, Carus Mathematics Monograph No. 13, Math. Assoc. Amer., Wiley, New York, 1960, pp. 154–156.

3. N. Bourbaki, *Espaces vectoriels topologiques*, Actualités Sci. Indust., No. 1189, Hermann, Paris, 1953; p. 46.

4. J. Dieudonné, *Foundations of modern analysis*, Academic Press, New York, 1960; pp. 143ff.

5. D. V. Widder, *The Laplace transform*, Princeton Univ. Press, Princeton, N. J., 1941, pp. 144–147.

6. E. F. Wittlesey, *Analytic functions in Banach spaces*, Proc. Amer. Math. Soc. **16** (1965), 1077–1083.

Northwestern University