ON THE BOREL FIELDS OF A FINITE SET

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1. **Introduction.** Let X be a nonempty set, Σ the complete lattice of all distinct topologies on X, and Δ the complete lattice of all Borel fields on X. It is now known [**3**] that Σ is a complemented lattice. If Borel field B is a topological Borel field, say generated by topology T, we shall indicate this by writing $B[T]$.

Consider the map $\Sigma \to \Delta$ given by $T \to B[T]$. We shall say topologies T_1 and T_2 are "Borel-equivalent," $T_1 \sim T_2$, iff $B[T_1] = B[T_2]$. Then $\sim$ is an equivalence relation, so we may speak of the quotient space $\Sigma/\sim$. If X is a countable space, the map is onto. If X is finite, Δ is a complemented sublattice of Σ. The structure of $\Sigma/\sim$ is examined for the cases $\text{card}(X) = 2$ and 3, and the structure of Δ for $\text{card}(X) = 4$. The method of K-matrices [**2**] is developed as a useful approach to the finite case.

2. **Development.**

Lemma. *If X is a countable space and B a nonempty family of subsets of X, the following are equivalent*:

(a) *B is a Borel field.*

(b) *B is a closed-open topology.*

(c) *B is a topological Borel field.*

Proof. Easily, every Borel field on a countable space is a topology, and being closed under complementation, each of its sets is closed-open. On the other hand, a closed-open topology is closed under complementation, so it is a Borel field.

If $\text{card}(X) = n$, (finite), let $f(n)$ be the number of distinct topologies that can be defined on X. Let $g(n)$ be the number of distinct Borel fields that can be defined on X.

Corollary. *If $1 < n$, then $g(n)/f(n) < 1$.*

It can readily be found that $g(1) = 1$, $g(2) = 2$, $g(3) = 5$, and $g(4) = 15$. Krishnamurthy [2] obtained the following values for f: $f(1) = 1$, $f(2) = 4$, $f(3) = 29$, $f(4) = 355$. As an upper bound on $f(n)$, he finds that $f(n) \leqq 2^{n(n-1)}$. Below, we shall note that the square root of this number is an upper bound on $g(n)$. Krishnamurthy uses what we shall here call "K-matrices."

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Let $\operatorname{card}(X)=n$. A "K-matrix" is an $n\times n$ matrix (a_{ij}) where for all i, j, $a_{ij}=0$ or 1, for all i, $a_{ii}=1$, and for all i, j: $(a_{ji}=1)$ implies $[(a_{ik}=1)$ implies $(a_{jk}=1)]$.

A correspondence between the set of all $n\times n$ K-matrices and the family of all distinct topologies on X is obtained by taking the ith row $R_i=S(x_i)$ as the smallest open neighborhood of x_i, determined by $a_{ij}=1$ iff $x_j\in S(x_i)$. The final condition on K-matrices then reads: if $x_i\in S(x_j)$, then $S(x_i)\subseteq S(x_j)$. It is a standard exercise [**1**, Chapter 1, Exercise B] that the filters generated by the $S(x_i)$ subject to this condition, correspond to a unique topology and that this correspondence is 1-1 and onto. Hereafter, we shall often identify a topology T with its K-matrix (a_{ij}).

THEOREM. *Let topology T have K-matrix (a_{ij}). Then T is a closed-open topology iff (a_{ij}) is symmetric (with respect to the main diagonal).*

PROOF (ONLY IF). Suppose for some i, j, $a_{ij}=1$ and $a_{ji}=0$. Then $x_j\in S(x_i)$ and $x_i\notin S(x_j)$. Hence $S(x_j)$ is a proper subset of $S(x_i)$. Suppose T is closed-open. Then $S(x_j)$ is closed, so $S(x_i)\backslash S(x_j)$ is open and $x_i\in S(x_i)\backslash S(x_j)$. But this is a proper subset of $S(x_i)$, contradicting $S(x_i)$ the smallest open neighborhood of x_i.

(IF). Let (a_{ij}) be a symmetric K-matrix and suppose some set t in its topology, T, is not closed. Then $\emptyset\neq X\backslash t$ is not open, so there exists an $x_j\in X\backslash t$ such that $S(x_j)$ is not a subset of $X\backslash t$. Hence $S(x_j)\cap t\neq\emptyset$. Let $x_i\in S(x_j)\cap t$ (open), so $S(x_i)\subseteq S(x_j)\cap t\subseteq S(x_j)$. Then $x_i\in S(x_j)$, so $a_{ji}=1$. Yet $x_j\notin S(x_i)$, so $a_{ij}=0$. Contradiction.

COROLLARY. *If $1\leqq n$, then $g(n)\leqq 2^{n(n-1)/2}$.*

For there are that many symmetric $n\times n$ matrices whose entries are 0 and 1, and which have 1's on the main diagonal. It is conjectured that $\lim_{n\to\infty} g(n)/f(n)=0$.

LEMMA. *Let (a_{ij}) be a K-matrix and define $b_{ij}=1$ iff $a_{ij}=a_{ji}=1$. Then (b_{ij}) is the Borel field generated by the topology (a_{ij}).*

PROOF. First, (b_{ij}) is a K-matrix. Suppose for some distinct i, j, $b_{ij}=1$. Then $a_{ij}=a_{ji}=1$. Claim: if for any k, $b_{jk}=1$, then $b_{ik}=1$. But if $b_{jk}=1$, then $a_{jk}=a_{kj}=1$. Now $a_{ij}=1$ and $a_{jk}=1$, so $a_{ik}=1$. Moreover $a_{kj}=1$ and $a_{ji}=1$, so $a_{ki}=1$. Thus $a_{ik}=a_{ki}=1$, and $b_{ik}=1$.

Now let T_1 and T_2 have K-matrices (a_{ij}) and (b_{ij}) respectively. It is easily checked that $T_1\subseteq T_2$ iff whenever $b_{ij}=1$, then $a_{ij}=1$. (b_{ij}) is symmetric by construction, so T_2 is a Borel field and is clearly the smallest Borel field containing T_1.

LEMMA. *Topology (a_{ij}) generates the Borel field $P(X)$, the power set, iff for all distinct i, j, row $R_i \neq$ row R_j.*

PROOF (ONLY IF). Suppose there exists some k, p such that $k \neq p$ and $R_k = R_p$. Then $a_{kp} = a_{pk} = 1$. Now let (b_{ij}) be given by: for all i, $b_{ii} = 1$, $b_{kp} = b_{pk} = 1$, and $b_{ij} = 0$ otherwise. Then (b_{ij}) is a proper Borel field and $(a_{ij}) \subseteq (b_{ij})$.

(IF). Suppose for all distinct i, j, $R_i \neq R_j$. If $a_{ij} = 1$, then $a_{ji} = 0$, else $S(x_i) = S(x_j)$ and $R_i = R_j$. Hence to find the generated Borel field, let $b_{ij} = 0$ whenever $i \neq j$. But this gives the identity matrix, hence the power set.

THEOREM. *Let (a_{ij}) be a K-matrix. Then $|a_{ij}| = 0$ or 1, and $|a_{ij}| = 1$ iff (a_{ij}) generates Borel field $P(X)$.*

PROOF. The process of reducing the matrix to find its generated Borel field shows the matrix to be row equivalent to the identity matrix, and row equivalent 0-1 matrices have the same determinant.

COROLLARY. *Let B be a Borel field with $n \times n$ K-matrix (a_{ij}). If B contains $n-1$ singletons, then $B = P(X)$.*

PROOF. If B contains $n-1$ singletons, then $|a_{ij}| = 1$.

3. **Lattice structure of Δ.** It is clear that the intersection of two closed-open topologies is a closed-open topology. Hence if X is countable, the Σ-meet and the Δ-meet of any two Borel fields coincide. On the other hand, the smallest topology containing the union of any two Borel fields is contained in the smallest closed-open topology containing that union. To see that at least in the finite case, the Σ-join coincides with the Δ-join, observe that the identification of topologies with their K-matrices induces a lattice structure on the K-matrices.

A handy observation for the Σ-join of two K-matrices is $(c_{ij}) = (a_{ij}) \vee (b_{ij})$ iff $[(c_{ij} = 1)$ iff $(a_{ij} = 1)$ and $(b_{ij} = 1)]$. Correspondingly for the Σ-meet of K-matrices, $(d_{ij}) = (a_{ij}) \wedge (b_{ij})$ iff (d_{ij}) is the smallest (least number of 1's) K-matrix containing (e_{ij}), where $(e_{ij} = 1)$ iff $(a_{ij} = 1)$ or $(b_{ij} = 1)$.

THEOREM. *If X is finite, then Δ is a sublattice of Σ.*

PROOF. Let B_1 and B_2 be Borel fields with K-matrices (a_{ij}) and (b_{ij}) respectively. Let $(c_{ij}) = (a_{ij}) \vee (b_{ij})$. But by the construction of (c_{ij}), since (a_{ij}) and (b_{ij}) are both symmetric, so is (c_{ij}). Hence the Σ-join of Borel fields is a Borel field.

THEOREM. *If X is finite, then Δ is a complemented lattice.*

PROOF. Let (b_{ij}) be a proper Borel field. By a previous result, if (b_{ij}) contains $n-1$ singletons, it contains n singletons. Hence let the number of singletons of (b_{ij}) be k, and note $0 \leqq k \leqq n-2$. Let $\{x_i\}$ be a singleton not in (b_{ij}) and consider the Borel field $M_1 = \{\varnothing, X, \{x_i\}, X\setminus\{x_i\}\}$. Clearly $M_1 \wedge (b_{ij}) = \{\varnothing, X\}$. Suppose $M_1 \vee (b_{ij}) \neq (\delta_{ij})$, the identity matrix. Let p stand for the number of singletons in $M_1 \vee (b_{ij})$ and notice $k < p \leqq n-2$. Let $\{x_j\}$ be a singleton not in $M_1 \vee (b_{ij})$ and let M_2 be the Borel field generated by the pair of singletons $\{x_i\}$, $\{x_j\}$. Claim: $M_2 \wedge (b_{ij}) = \{\varnothing, X\}$. [For if $\{x_i, x_j\} \in (b_{ij})$, then $\{x_i, x_j\}\setminus\{x_i\} = \{x_j\} \in M_1 \vee (b_{ij})$.] If $M_2 \vee (b_{ij}) \neq (\delta_{ij})$, repeat the process. At each step, $M_k \wedge (b_{ij}) = \{\varnothing, X\}$, and for some m we must have $M_m \vee (b_{ij}) = (\delta_{ij})$.

It follows from a result of Steiner's [**3**, Theorem 1.2] that a Borel field on a countable space is a principle topology. Since Steiner establishes that every topology has a principle complement, it seems likely that Δ is a complemented lattice whenever X is countable.

4. **Examples.** For a space of two points, the two Sierpinski (proper) topologies are both Borel-equivalent to the power set.

As an application of this, consider the following well-known result. If (X, T_1) and (Y, T_2) are arbitrary topological spaces, and if f: $(X, T_1) \to (Y, T_2)$ is a homeomorphism, then f: $(X, B[T_1]) \to (Y, B[T_2])$ is a Baire function. A counterexample to the converse can be found on a space of two points. Let S_1 and S_2 be the two Sierpinski topologies on X. Then $B[S_1] = B[S_2] = P(X)$. The identity map id: $(X, B[S_1]) \to (X, B[S_2])$ is a Baire function, yet id: $(X, S_1) \to (X, S_2)$ is not continuous.

We shall use the "n-basic number" notation of [2] to look at the lattice structure of $\Sigma/\sim$ for the case $n=3$, and that of Δ for $n=4$. The "n-basic number" for an $n \times n$ K-matrix is the integer obtained by suppressing the main diagonal of the matrix and writing the remaining entries, in lexographic order of indices, as a binary number. Similarly given an n-basic number, its K-matrix can be reconstructed.

For a space of three points, the 3-basic numbers which give Borel fields are 5, 18, 40, 63, and 64. Now 63 $[\{\varnothing, X\}]$ and 64 $[P(X)]$ are complementary, of course. The proper Borel fields 5, 18, and 40 are pairwise complementary. Moreover, the topologies 43 and 60 are Borel-equivalent to 40, the topologies 30 and 51 to 18, and topologies 15 and 53 to 5. All the others (except 63) are Borel-equivalent to 64.

For a space of four points, the 4-basic numbers for Borel fields and for their corresponding complements are given in the following table:

Borel field	Is a complement of (in Δ)
9	660, 1122, 2886, 3504
66	660, 1581, 2313, 3504
144	1122, 1581, 2313, 2886
219	516, 1056, 2304
516	219, 1122, 2313, 3504
660	9, 66, 1056, 1122, 2304, 2313
1056	219, 660, 2313, 2886
1122	9, 144, 516, 660, 2304, 2313
1581	66, 144, 2304
2304	219, 660, 219, 1122, 1581
2313	66, 144, 516, 660, 1056, 1122
2886	9, 144, 1056
3504	9, 66, 516
4095 [$\{\phi, X\}$]	4096
4096 [$P(X)$]	4095

5. **Observations.** It is of some interest to note that "is a complement of" is in general neither unique nor transitive. It would be of further interest to know, in a finite space, the minimum number of complements (in Δ) a proper Borel field can have as a function of n, and the minimum number of topologies in the proper equivalence classes of $\Sigma/\sim$. This latter could be used as a measure of how badly many-one the map $\Sigma \rightarrow \Delta$ is, and could give information toward our conjecture on $\lim_{n\to\infty} g(n)/f(n)$.

We have shown a characterization of those topologies Borel-equivalent to $P(X)$, namely that their K-matrices be nonsingular. A study of those topologies ("sparse" topologies) whose generated Borel fields are proper will be made in a later paper.

References

1. J. L. Kelley, *General topology*, Van Nostrand, New York, 1955.

2. V. Krishnamurthy, *On the number of topologies on a finite set*, Amer. Math. Monthly **73** (1966), 154–157.

3. A. K. Steiner, *The topological complementation problem*, Bull. Amer. Math. Soc. **72** (1966), 125–127.

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