A COUNTEREXAMPLE TO A THEOREM OF CHASE

PHILLIP GRIFFITH[1]

1. **Introduction.** Throughout this note all groups are abelian. The notation is that of [2]. If G is a torsion free group and H is a subgroup of G, we shall use the symbol H_* to denote the minimal pure subgroup of G containing H. Also, the symbols $\sum$ and $+$ will be used for direct sums; whereas the subgroup of a group G generated by its subsets S and T will be denoted by $\{S, T\}$.

In answer to a question of Nunke, Chase [1] proved incorrectly the following theorem and its immediate corollary.

THEOREM (CHASE). *Let G be a torsion free group, and let F_0 and F_1 be pure free subgroups of G of rank α and β respectively where $\alpha<\beta$ and β is infinite. Then there exists a pure free subgroup F of G such that F_0 is a direct summand of F and rank $(F)=\beta$.*

COROLLARY (CHASE). *If X_0 and X_1 are maximal pure independent subsets of a torsion free group G, one of which is infinite, then both have equal cardinality.*

In fact, these results are false as stated. We shall demonstrate this by exhibiting a counterexample to the corollary. Chase's proof is correct when α and β are both infinite, but there is a flaw when α is finite and β is infinite. We construct a countable torsion free group G with maximal pure independent subsets X_0 and X_1 such that $|X_0|=1$ and $|X_1|=\aleph_0$. Letting F_0 and F_1 be the pure free subgroups generated by X_0 and X_1 respectively, we immediately observe that Chase's theorem and corollary do not hold for G. Finally, we remark that a careful study of Chase's proof yields that counterexamples to the theorem can occur only when α is finite and β is at most countable.

2. **Counterexample.** We form the direct sum $A=Q+D$ where Q denotes the rational numbers and D is the minimal divisible group containing a free group F of rank $\aleph_0$. Let $p_1<p_2<\cdots$ be the usual ordering of the primes. Define $[b_1, b_2, \cdots]$ to be the collection of all nonzero elements x of F such that x generates a pure cyclic subgroup of F. Note that no b_n is divisible in F by any prime. Select a nonzero element y in Q and set $B=[p_n^{-k}(y+b_n) \mid n=1, 2, \cdots$ and $k=1, 2, \cdots]$. Let G be the subgroup of A generated by the group

Received by the editors March 13, 1967.

[1] The author wishes to acknowledge support by the National Aeronautics and Space Administration Grant NsG(T)—52.

$\{y\}+F$ and the set B. We show that $\{y\}$ and F are pure subgroups of G.

Observe that a typical element $g\in G$ has the form

$$g = f + ty + \sum_{i=1}^{r} m_i(y + b_i),$$

where $f\in F$, t is an integer, $m_i=t_ip_i^{-k_i}$, and t_i is an integer such that $(t_i, p_i)=1$ for $i=1, \cdots, r$. If $ng\in\{y\}$ for some nonzero integer n, then $nf+n\sum_{i=1}^r m_ib_i=0$ since $A=Q+D$. Therefore, $f=-\sum_{i=1}^r m_ib_i$. Set $u_i=\prod_{e=1;e\neq i}^r p_e^{k_e}$. It follows that $(p_i, u_i)=1$ and that u_im_e is an integer for $i\neq e$. Therefore, $-u_im_ib_i=u_if+\sum_{e=1;e\neq i}^r u_im_eb_e\in F$. Our definition of b_i implies that u_im_i must be an integer for $i=1, \cdots, r$. Since $(p_i, u_i)=1$ and since $m_i=t_ip_i^{-k_i}$, it follows that u_im_i is an integer if and only if m_i is an integer for $i=1, \cdots, r$. Hence, $g=(t+\sum_{i=1}^r m_i)y\in\{y\}$. Thus, $\{y\}$ is a pure subgroup of G. If $ng\in F$, then $nty=(-n\sum_{i=1}^r m_i)y$ since $A=Q+D$. Therefore, $ty=(-\sum_{i=1}^r m_i)y$ implies that $-\sum_{i=1}^r m_i$ is an integer. Hence, as observed above, this implies that each m_i is an integer. Therefore, $g=f+\sum_{i=1}^r m_ib_i\in F$ and F is a pure subgroup of G.

Expressing F as $F=\sum_{i<\omega}\{x_i\}$, we have that $[x_i, x_2, \cdots]$ is a pure independent subset of G, since F is a pure subgroup of G. Let X_1 be a maximal pure independent subset of G containing $[x_1, x_2, \cdots]$. Then $|X_1|=\aleph_0$. Since $\{y\}$ is a pure subgroup of G, $X_0=[y]$ is a pure independent subset of G. We establish that X_0 is a maximal pure independent subset of G by showing that $\{y, g\}_*$ is not free for any $g\in G$ which is independent from y.

Let $g\in G$ be such that g and y are independent. Since $A/(\{y\}+F)$ is torsion, there is a nonzero integer n and an integer m such that $ng=my+f$ where $f\in F$. The independence of y and g implies that $f\neq 0$. Therefore $ng-my=f\in\{y, g\}_*$. For some positive integers u and v, we have that $ub_v=f$. Therefore b_v must be an element of $\{y, g\}_*$. Since $y+b_v\in\{y, g\}_*$ and since $\{y, g\}_*$ is a pure subgroup of G, then $p_v^{-k}(y+b_v)\in\{y, g\}_*$ for $k=1, 2, \cdots$. Thus $\{y, g\}_*$ cannot be free.

References

1. S. Chase, "On group extensions and a problem of J. H. C. Whitehead" pp. 173–193, in *Topics in abelian groups*, Scott, Foresman and Co., Chicago, Ill., 1963.

2. L. Fuchs, *Abelian groups*, Akad. Kiadó, Budapest, 1958.

University of Houston