BLASCHKE QUOTIENTS AND NORMALITY

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Let $f$ be meromorphic in the unit disc $D$ and call $f$ normal if the family $\{f \circ S_n\}$ is normal, where $\{S_n\}$ represents the family of one to one conformal mappings of $D$ onto $D$. Further, if $\{z_n\}$ is a sequence in $D$ and $w = \{w_1, w_2, w_3, \cdots\}$ is a vector in $l^\infty$ we say that $\{z_n\}$ interpolates if there is a function in $H^\infty$ with $f(z_n) = w_n$, $n = 1, 2, 3, \cdots$. A criterion that $\{z_n\}$ interpolate was given by L. Carleson [2] as

$$A(n) = \prod_{k=1; k \neq n}^{\infty} \left| \frac{(z_n - z_k)/(1 - \bar{z}_k z_n)}{\delta} \right| \geq \delta > 0,$$

all $n = 1, 2, 3, \cdots$. Our first result states that if $\{z_n\}$ interpolates then the Blaschke product for $\{z_n\}$, written as $B(z; z_n)$, has its modulus greater than some positive number $\eta$ for points not close to $\{z_n\}$.

More precisely we let $\psi(z, z_n) = |(z-z_n)/(1-\bar{z}_nz_n)|$ be the pseudo-hyperbolic distance in the disc of $z$ to $z_n$. Then we have

**Theorem 1.** If $\{z_n\}$ interpolates and $B(z) = B(z; z_n)$ is the Blaschke product for $\{z_n\}$ then for each $\epsilon > 0$ there is a $\eta > 0$ such that $|B(z)| \geq \eta$ whenever $\psi(z, z_n) \leq \epsilon$, $n = 1, 2, 3, \cdots$.

**Proof.** Assume there is a sequence $\{t_k\}$ in $D$ and $\epsilon_0 > 0$ so that $\psi(t_k, z_n) \geq \epsilon_0$ all $k$ and $n$ and that $B(t_k)$ tends to zero as $k$ tends to infinity. By a result of A. T. Cargo [1, p. 142] we may select a subsequence of $\{t_k\}$, which we again write as $\{t_k\}$, so that the Blaschke product $A(z) = A(z; t_k)$ formed from the sequence $\{t_k\}$ has the property that

$$|A(z_n)| \geq a > 0, \quad n = 1, 2, 3, \cdots.$$

We have then

$$|A(z)| + |B(z)| > 0, \quad z \in D.$$

Then as in Theorem 1 of [3] we can find functions $g_1(z)$ and $g_2(z)$ holomorphic in $D$ and satisfying

$$A(z)g_1(z) + B(z)g_2(z) = 1, \quad z \in D.$$

The sequence $\{(A(z_n))^{-1}\}$ is in $l^\infty$ so there is an $f_1$ in $H^\infty$ satisfying $f_1(z_n) = (A(z_n))^{-1}$, $n = 1, 2, 3, \cdots$. Also we have that

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796
(f_1(z) - g_1(z)) \cdot (B(z))^{-1} = h(z)

with \( h \) holomorphic in \( D \).

Define a holomorphic function \( f_2 \) on \( D \) as follows:

\[
f_2(z) = g_2(z) - h(z)A(z).
\]

Then

\[
f_1(z)A(z) + f_2(z)B(z) = 1, \quad z \in D,
\]

and

\[
f_2(z) = (1 - f_1(z)A(z)) \cdot (B(z))^{-1}
\]

is a bounded holomorphic function. Evaluating \( f_2 \) on the sequence \( \{t_n\} \) shows it is unbounded, which is a contradiction, and so we have the result.

If we are given sequences of distinct points \( A_1 = \{\alpha_n\} \) and \( A_2 = \{\beta_n\} \) in \( D \) with corresponding Blaschke products \( B_1(z) = B_1(z; \alpha_n) \) and \( B_2(z) = B_2(z; \beta_n) \), we have the following theorem concerning the normality of the quotient \( B_1(z) \cdot (B_2(z))^{-1} \).

**Theorem 2.** If \( A_1 \) and \( A_2 \) are disjoint interpolating sequences in \( D \), then the meromorphic function \( f(z) = B_1(z) \cdot (B_2(z))^{-1} \) is normal in \( D \) if and only if \( A_1 \cup A_2 \) interpolates.

**Proof.** Assume first \( f(z) = B_1(z) \cdot (B_2(z))^{-1} \) is normal in \( D \). A criterion of O. Lehto and K. I. Virtanen [4, pp. 55-56] states that

\[
\rho(f) \leq C \left| \frac{dz}{(1 - |z|^2)} \right|, \quad |z| < 1,
\]

where \( \rho \) is the spherical derivative of \( f \) and \( C \) is a positive constant. Evaluating this inequality on the points \( \{\alpha_i\} \) of \( A_1 \) yields

\[
|B_2(\alpha_i)| \geq (1/C) |B_1(\alpha_i)| (1 - |\alpha_i|^2).
\]

We denote the partial product

\[
\prod_{j \neq l} \frac{1 - \overline{\alpha}_j z}{z - \overline{\alpha}_j} B_1(z) = \prod_{n=1; n \neq j}^\infty \frac{-\alpha_n}{|\alpha_n|} \left( \frac{z - \alpha_n}{1 - \overline{\alpha}_n z} \right)
\]

as \( \prod_j(z) \) and write \( B_1(z) \) as

\[
B_1(z) = \frac{-\alpha_j}{|\alpha_j|} \left( \frac{z - \alpha_j}{1 - \overline{\alpha}_j z} \right) \cdot \prod_j(z).
\]

The condition of L. Carleson quoted previously shows that \( |\prod_j(\alpha_j)| \geq \delta > 0 \). This shows that

\[
\lim \inf |B_2(\alpha_j)| \geq (1/C)\delta > 0.
\]

Thus \( A_1 \cup A_2 \) interpolates.
Now assume \( A_1 \cup A_2 \) interpolates. If the set of numbers

\[
|f'(z)| \frac{(1 - |z|^2)}{(1 + |f(z)|^2)}
\]

were unbounded there must exist a sequence \( \{z_k\} \) in \( D \) such that

\[
|B_1(z_k)|^2 + |B_2(z_k)|^2 \to 0.
\]
as \( k \) tends to infinity. By Theorem 1 it must be the case that for some subsequence of \( A_1 \), say \( \{\alpha_{k_n}\} \), we have \( \psi(\alpha_{k_n}, \alpha_{k_n}) \) tends to zero. Also there must be a subsequence \( \{\beta_{j_k}\} \) with \( \psi(\beta_{j_k}, \beta_{j_k}) \) tending to zero. This shows that \( \psi(\alpha_{j_k}, \beta_{j_k}) \) tends to zero which is a contradiction to \( A_1 \cup A_2 \) interpolating. The spherical derivative is bounded by a constant times \( (1 - |z|^2)^{-1} |dz| \) and so \( f \) is normal.

We point out that \( \rho(f) = \rho(1/f) \) so that our Theorem 2 can be stated for \( B_2(z) \cdot (B_1(z))^{-1} \). The referee has pointed out the following equivalence.

**Theorem 3.** If \( A_1 \) and \( A_2 \) are disjoint interpolating sequences in \( D \), then \( B_1(z) \cdot (B_2(z))^{-1} \) is normal if and only if \( B_1 \) and \( B_2 \) are in no proper ideal of \( H^\infty \).

**Proof.** If the ideal generated by \( B_1 \) and \( B_2 \) is not proper there are functions \( f \) and \( g \) in \( H^\infty \) satisfying

\[
f(z)B_1(z) + g(z)B_2(z) = 1, \quad z \in D.
\]

Setting \( h_1(z) = f(z)B_1(z) \) and \( h_2(z) = g(z)B_2(z) \) we observe that \( h_1 \) is zero on \( A_1 \) and one on \( A_2 \). Similarly \( h_2 \) is one on \( A_1 \) and zero on \( A_2 \). Using these functions and the hypothesis that \( A_1 \) and \( A_2 \) interpolate, we see that \( A_1 \cup A_2 \) interpolates.

Conversely, if \( A_1 \cup A_2 \) is interpolating then there is an \( f \in H^\infty \) with \( fB_1 = 1 \) on \( A_2 \). Hence \( 1 - fB_1 \) is divisible by \( B_2 \) and thus one is in the ideal generated by \( B_1 \) and \( B_2 \).

**References**

3. J. A. Cima and G. D. Taylor, *On the equation \( f_1 g_1 + f_2 g_2 = 1 \) in \( H^\infty \),* Illinois J. Math. (to appear).

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