A STRONG COMPARISON THEOREM FOR SELFADJOINT ELLIPTIC EQUATIONS

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The purpose of this note is to give a concise proof of a comparison theorem for selfadjoint, second order elliptic equations which yields stronger results than those previously derived in [1], [2] and [3]. All coefficients and domains are to be sufficiently smooth so that the variational techniques of Courant [4] can be applied. Specifically, it is assumed that the first eigenfunction of the selfadjoint boundary value problem

$$(1)\qquad \begin{aligned} &-\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(\alpha_{ij}\frac{\partial v}{\partial x_i}\right) + \gamma v = \lambda v \quad \text{in } D,\\ &\frac{\partial v}{\partial \nu} + \sigma v = 0 \quad \text{on } \partial D, \quad -\infty < \sigma(x) \leqq +\infty \end{aligned}$$

can be determined uniquely (up to a multiplicative constant) by minimizing

$$(2)\qquad \mathfrak{D}[\phi] = \int_D \left[\sum \alpha_{ij}\frac{\partial \phi}{\partial x_i}\frac{\partial \phi}{\partial x_j} + \gamma\phi^2\right]dx + \int_{\partial D} \sigma\phi^2 d\bar{x}$$

over all "admissible" $\phi \in \Phi$. The class Φ consists of all real valued functions which are continuous in $\overline{D}$ have piecewise continuous first partials in D, vanish on $\{x \in \partial D \mid \sigma(x) = +\infty\}$ and satisfy $\int_D \phi^2 dx = 1$. (Here, $\sigma(x) = +\infty$ is used to denote the boundary condition $v = 0$.) It is further assumed that all coefficients and D are sufficiently regular so that this extremal function is a solution of (1) in the classical sense.

THEOREM. *Suppose $u(x)$ and $v(x)$ are solutions respectively of*

$$(3)\qquad \sum \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial u}{\partial x_i}\right) = cu,$$

$$(4)\qquad \sum \frac{\partial}{\partial x_j}\left(\alpha_{ij}\frac{\partial v}{\partial x_i}\right) = \gamma v$$

in a domain $G \supset \overline{D}$ and that

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$$\int_D \left[\sum a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + cu^2 \right] dx \geqq \int_D \left[\sum \alpha_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + \gamma u^2 \right] dx.$$

If

$$\partial u/\partial \nu + s(x)u = 0 \quad \text{on } \partial D \tag{5}$$

and

$$\partial v/\partial \nu + \sigma(x)v = 0 \quad \text{on } \partial D \tag{6}$$

with $-\infty < \sigma(x) \leqq s(x) \leqq +\infty$, *then either* $v(x)$ *has a zero in the interior of* D *or else* u *is a constant multiple of* v.

PROOF. Let $B_1 = \{x \in \partial D \mid \sigma(x) < \infty\}$ and $B_2 = \{x \in \partial D \mid s(x) < \infty\}$. Without loss of generality we may assume $\int_D u^2 dx = 1$ so that u is admissible with respect to the variational problem (2). If $v(x) \neq 0$ in the interior of D, then v is the first eigenfunction of (1) corresponding to the eigenvalue $\lambda_1 = 0$. Therefore

$$\begin{aligned} 0 &= \inf_{\phi \in \Phi} \int_D \left[\sum \alpha_{ij} \frac{\partial \phi}{\partial x_i} \frac{\partial \phi}{\partial x_j} + \gamma \phi^2 \right] dx + \int_{B_1} \sigma \phi^2 d\bar{x} \\ &\leqq \int_D \left[\sum \alpha_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + \gamma u^2 \right] dx + \int_{B_1} \sigma u^2 d\bar{x} \\ &\leqq \int_D \left[\sum a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + cu^2 \right] dx + \int_{B_2} s u^2 d\bar{x}. \end{aligned}$$

However in view of (3), (5) and Green's theorem, the last term is zero so that we have equality throughout the above expression. In particular, we see that $u(x)$ is an extremal function for the variational problem (2) and therefore an eigenfunction of (1) corresponding to $\lambda_1 = 0$. In light of the simplicity of the first eigenvalue of (1), u is a constant multiple of v.

Setting $s(x) \equiv +\infty$ on ∂D, we obtain a stronger form of the comparison theorem derived in [**3**].

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