A STRONG COMPARISON THEOREM FOR SELFADJOINT ELLIPTIC EQUATIONS

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The purpose of this note is to give a concise proof of a comparison theorem for selfadjoint, second order elliptic equations which yields stronger results than those previously derived in [1], [2] and [3]. All coefficients and domains are to be sufficiently smooth so that the variational techniques of Courant [4] can be applied. Specifically, it is assumed that the first eigenfunction of the selfadjoint boundary value problem

\[- \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j} (\alpha_{ij} \frac{\partial v}{\partial x_i}) + \gamma v = \lambda v \quad \text{in } D,\]

\[\frac{\partial v}{\partial \nu} + \sigma v = 0 \quad \text{on } \partial D, \quad -\infty < \sigma(x) \leq +\infty\]

can be determined uniquely (up to a multiplicative constant) by minimizing

\[\mathcal{D}[\phi] = \int_{D} \left[ \sum \alpha_{ij} \frac{\partial \phi}{\partial x_i} \frac{\partial \phi}{\partial x_j} + \gamma \phi^2 \right] dx + \int_{\partial D} \sigma \phi^2 dx\]

over all "admissible" \(\phi \in \Phi\). The class \(\Phi\) consists of all real valued functions which are continuous in \(D\) have piecewise continuous first partials in \(D\), vanish on \(\{x \in \partial D : \sigma(x) = +\infty\}\) and satisfy \(\int_{D} \phi^2 dx = 1\). (Here, \(\sigma(x) = +\infty\) is used to denote the boundary condition \(v = 0\).) It is further assumed that all coefficients and \(D\) are sufficiently regular so that this extremal function is a solution of (1) in the classical sense.

**Theorem.** Suppose \(u(x)\) and \(v(x)\) are solutions respectively of

\[\sum \frac{\partial}{\partial x_j} \left( \alpha_{ij} \frac{\partial u}{\partial x_i} \right) = cu,\]

\[\sum \frac{\partial}{\partial x_j} \left( \alpha_{ij} \frac{\partial v}{\partial x_i} \right) = \gamma v\]

in a domain \(\Omega \supset D\) and that

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\[
\int_D \left[ \sum a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + cu^2 \right] dx = \int_D \left[ \sum \alpha_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + \gamma u^2 \right] dx.
\]

If
\[
(5) \quad \frac{\partial u}{\partial v} + s(x)u = 0 \quad \text{on } \partial D
\]

and
\[
(6) \quad \frac{\partial v}{\partial v} + \sigma(x)v = 0 \quad \text{on } \partial D
\]

with \(-\infty < \sigma(x) \leq s(x) \leq +\infty\), then either \(v(x)\) has a zero in the interior of \(D\) or else \(u\) is a constant multiple of \(v\).

**Proof.** Let \(B_1 = \{ x \in \partial D \mid \sigma(x) < \infty \} \) and \(B_2 = \{ x \in \partial D \mid s(x) < \infty \} \). Without loss of generality we may assume \(\int_D u^2 dx = 1\) so that \(u\) is admissible with respect to the variational problem (2). If \(v(x) \neq 0\) in the interior of \(D\), then \(v\) is the first eigenfunction of (1) corresponding to the eigenvalue \(\lambda_1 = 0\). Therefore

\[
0 = \inf_{\phi \in \Phi} \int_D \left[ \sum \alpha_{ij} \frac{\partial \phi}{\partial x_i} \frac{\partial \phi}{\partial x_j} + \gamma \phi^2 \right] dx + \int_{B_1} \sigma \phi^2 dx
\]

\[
\leq \int_D \left[ \sum \alpha_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + \gamma u^2 \right] dx + \int_{B_1} \sigma u^2 dx
\]

\[
\leq \int_D \left[ \sum a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + cu^2 \right] dx + \int_{B_2} s u^2 dx.
\]

However in view of (3), (5) and Green's theorem, the last term is zero so that we have equality throughout the above expression. In particular, we see that \(u(x)\) is an extremal function for the variational problem (2) and therefore an eigenfunction of (1) corresponding to \(\lambda_1 = 0\). In light of the simplicity of the first eigenvalue of (1), \(u\) is a constant multiple of \(v\).

Setting \(s(x) \equiv +\infty\) on \(\partial D\), we obtain a stronger form of the comparison theorem derived in [3].

**Bibliography**


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