ON THE EXISTENCE OF RIGID COMPACT ORDERED SPACES

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1. It is easily seen that every compact ordered space with infinitely many points which has a countable base admits continuously many autohomeomorphisms. For, if there are countably many isolated points, the assertion is obvious. In the other case the assertion follows from the fact that there is either a separable connected subspace which is consequently homeomorphic to an interval of the real numbers or the space is zero-dimensional and so is homeomorphic to the Cantor set (possibly except for finitely many isolated points). Jónsson [1] and Rieger [2] each give an example of an infinite zero-dimensional compact ordered space S, which is rigid, i.e. a space such that the only homeomorphism of S onto S is the identity mapping. However, the weight of the constructed space S is very large. (Here weight means the minimal cardinality of an open base.) The purpose of this note is to show the existence of (a family of 2^c) rigid zero-dimensional compact ordered spaces each having continuous weight and continuous power. (Here c denotes the cardinal of the set of real numbers.)

2. If N is a set and f is a map of a subset S of N into N, then for every $M\subset N$ we write fM instead of $f[S\cap M]$.

In de Groot [1] the following concepts were introduced:

(a). If f is a map of $S\subset N$ into N, then f is called a *displacement* of *order m*, if there exists a subset V of N such that

$$V\cap fV=\varnothing,\qquad |fV|=m,$$

whereas for no $n>m$ there is a subset W of N, such that

$$W\cap fW=\varnothing,\qquad |fW|=n.$$

(b). If N is a topologicals pace, then f is called a *continuous displacement* of $S\subset N$, if f is a continuous map of S into N and f is a displacement of order c.

One easily proves the following generalization of [1, Lemma 2].

LEMMA. *Let P be a separable metric space. Let Q be a subset of P of which every point is a point of condensation in Q.*

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If ψ: $Q \rightarrow P$ is a map ($\neq$ identity) such that every point of $\psi[Q]$ is again a point of condensation in $\psi[Q]$, then ψ is a continuous displacement in P.

Strengthening in a trivial way the result of [**1**, Theorem 1], for the case that M = the unit interval I of the reals and $\{K_\beta\}$ is the system of all uncountable compact subsets of I (cf. [**1**, Theorem 2]), we obtain the following theorem.

THEOREM A. *There exists a family $\{F_\gamma\}$ of 2^c zero-dimensional subsets of I, each of power c, such that*

(i) *every subinterval of I contains c points of each F_γ and of each $I \backslash F_\gamma$ (in particular $\overline{F}_\gamma = I$ for all γ).*

(ii) $|F_\gamma \backslash F_{\gamma'}| = c$ *for all γ, γ' with $\gamma \neq \gamma'$.*

(iii) $|\phi F_\gamma \backslash F_{\gamma'}| = c$ *for all γ, γ' and all continuous displacements ϕ of F_γ in I.*

From this theorem it follows [**1**, Theorem 2], that there exists neither a proper autohomeomorphism of any F_γ, nor a homeomorphism of any F_γ onto some other $F_{\gamma'}$. (For, such a map would be a continuous displacement of F_γ (cf. [**1**, Lemma 2]), which is impossible according to Theorem A(iii).)

3. Define the set

$$S_\gamma = F_\gamma \cup \{(b, 0), (b, 1)\}_{b \in I \backslash F_\gamma}$$

and introduce an order in it in the natural way (i.e. if $\tilde{p} = p$ when $p \in F_\gamma$ and $\tilde{p} = b$ when $p = (b, 0)$ or $p = (b, 1)$, then $p < q$ in S if $\tilde{p} < \tilde{q}$ in I; and moreover $(b, 0) < (b, 1)$ for every $b \in I \backslash F_\gamma$). It is clear that S_γ (supplied with its interval topology) is a separable zero-dimensional compact space of weight c.

Every subspace of S_γ which contains c points of $S_\gamma \backslash F_\gamma$ has weight c. And F_γ-as-a-subspace-of-I is homeomorphic to F_γ-as-a-subspace-of-S_γ.

THEOREM B. *S_γ is rigid for every γ.*
S_γ is not homeomorphic to $S_{\gamma'}$, if $\gamma \neq \gamma'$.

PROOF. Let ϕ_γ be the continuous map of S_γ onto I which is defined by

$$\phi_\gamma(a) = a, \qquad \text{if } a \in F_\gamma,$$

$$\phi_\gamma((b, 0)) = \phi_\gamma((b, 1)) = b, \quad \text{if } b \in I \backslash F_\gamma.$$

Now suppose that f_γ is a proper autohomeomorphism of S_γ, and that

$f_{\gamma\gamma'}$ is a homeomorphism of S_γ onto $S_{\gamma'}$ $(\gamma \neq \gamma')$. Then it is easily checked (see Lemma) that

$$\psi_\gamma = (\phi_\gamma \cdot f_\gamma) \mid F_\gamma \quad \text{and} \quad \psi_{\gamma\gamma'} = (\phi_{\gamma'} \cdot f_{\gamma\gamma'}) \mid F_\gamma$$

are continuous displacements of F_γ in I. Consequently (Theorem A(iii)) $I \backslash F_\gamma$ contains c points of $\psi_\gamma F_\gamma$ and $I \backslash F_\gamma$ contains c points of $\psi_{\gamma\gamma'} F_\gamma$, and so $S_\gamma \backslash F_\gamma$ contains c points of $f_\gamma F_\gamma$ and $S_{\gamma'} \backslash F_{\gamma'}$ contains c points of $f_{\gamma\gamma'} F_\gamma$. However, this is a contradiction, since $f_\gamma F_\gamma$ and $f_{\gamma\gamma'} F_\gamma$ then would have weight c, whereas F_γ has weight $\aleph_0$.

COROLLARY. *If $B(X)$ denotes the Boolean algebra of the open-and-closed subsets of the zero-dimensional compact space X, then $\{B(S_\gamma)\}_\gamma$ is a family of 2^c Boolean algebras, each of power c, without proper automorphisms.*

REFERENCES

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