INTEGRAL REPRESENTATIONS AND 
REFINEMENT-UNBOUNDEDNESS

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1. Introduction. Suppose $U$ is a set, $F$ is a field of subsets of $U$, $p$ is the set of all real-valued functions defined on $F$, $p_A$ is the set of all bounded and finitely additive elements of $p$, $p^+$ is the set of all non-negative-valued elements of $p$, and $p_A^+ = p_A \cap p^+$.

Suppose $\mu$ is in $p_A^+$.

Definition. If $\mathfrak{M}$ is a number set and $\xi$ is in $p_A$, then the statement that $\xi$ is $\mu$-dense in $\mathfrak{M}$ means that if $V$ is in $F$ and $0 < c$, then there is a subdivision $\mathcal{E}$ of $V$ and a function $n$ from $\mathcal{E}$ into $\mathfrak{M}$ such that

$$\sum_{\mathcal{E}} | \xi(I) - n(I) \mu(I) | < c.$$

We prove the following integral representation theorem (§3):

Theorem 3.1. If $\mathfrak{M}$ is a bounded number set and $\xi$ is an element of $p_A$ which is $\mu$-dense in $\mathfrak{M}$, then there is a function $\theta$ from $F$ into $\mathfrak{M}$ (i.e., $\mathfrak{M}$ plus its closure) such that if $V$ is in $F$, then the integral (§2)

$$\int_V \theta(I) \mu(I)$$

exists and is $\xi(V)$.

The question naturally arises as to necessary and sufficient conditions under which, in the statement of Theorem 3.1, $\mathfrak{M}$ may be replaced by $\mathfrak{M}$. By considering the previously defined [1] notion of refinement-unboundedness (see [1] or §4 of this paper), we obtain the following characterization theorem (§4):

Theorem 4.1. The following three statements are equivalent:

1. If $\mathfrak{M}$ is a bounded number set and $\xi$ is an element of $p_A$ which is $\mu$-dense in $\mathfrak{M}$, then there is a function $\phi$ from $F$ into $\mathfrak{M}$ such that if $V$ is in $F$, then $\int_V \phi(I) \mu(I)$ exists and is $\xi(V)$.

2. If $\mathfrak{M}$ is a bounded number set and $\theta$ is a function from $F$ into $\mathfrak{M}$ and $\int \theta(I) \mu(I)$ exists, then there is a function $\phi$ from $F$ into $\mathfrak{M}$ such that if $V$ is in $F$, then $\int_V \phi(I) \mu(I)$ exists and is $\int \theta(I) \mu(I)$.

3. There is a $\mu$-refinement-unbounded (§4) element of $p^+$.

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2. Preliminary theorems and definitions. If $V$ is in $F$, then the statement that $\mathcal{D}$ is a subdivision of $V$ means that $\mathcal{D}$ is a finite collection of mutually exclusive sets of $F$ whose union is $V$.

If $\mathcal{D}$ is a subdivision of a set $V$ of $F$, then the statement that $\mathcal{E}$ is a refinement of $\mathcal{D}$ means that $\mathcal{E}$ is a subdivision of $V$, every set of which is a subset of some set of $\mathcal{D}$.

Throughout this paper all integrals considered will be Hellinger [3] type limits (i.e. for refinements of subdivisions) of the appropriate sums.

Suppose $\alpha$ is in $p$.

Suppose $\mathcal{D}$ is a subdivision of $U$. If $V$ is in $F$, then the statement that $\alpha$ is $\sum$-bounded on $V$ with respect to $\mathcal{D}$ means that if $\mathcal{R} = \{ z \mid z = \sum_{\mathcal{D}} \alpha(I), \mathcal{E}$ a subdivision of $V$ and a subset of a refinement of $\mathcal{D} \}$, then $-\infty < s^*(\alpha)(V) = \inf \mathcal{R} \leq \sup \mathcal{R} = s^*(\alpha)(V) < \infty$. We adopt the convention that throughout this paper, as in the preceding definition, $s^*$ and $s^*_*$ will be understood to be defined in terms of the last mentioned subdivision in the discussion at hand with respect to which the functions under consideration are $\sum$-bounded on $U$. We see that $\alpha$ is $\sum$-bounded on $U$ with respect to $\mathcal{D}$ iff for each $V$ in $F$, $\alpha$ is $\sum$-bounded on $V$ with respect to $\mathcal{D}$, in which case, if $V$ is in $F$ and $\mathcal{E}$ is a refinement of each of the subdivisions $\mathcal{D}$ and $\mathcal{D}'$ of $V$, then

$$\sum_{\mathcal{D}} s^*(\alpha)(I) \leq \sum_{\mathcal{E}} s^*(\alpha)(I) \leq \sum_{\mathcal{E}} s^*(\alpha)(I) \leq \sum_{\mathcal{D}'} s^*(\alpha)(I),$$

so that we have the existence and following relationship of the following integrals:

$$\int_V s^*(\alpha)(I) \leq \int_V s^*(\alpha)(I),$$

and we see that $\int_V \alpha(I)$ exists iff $\int_V s^*(\alpha)(I) = \int_V s^*(\alpha)(I)$, in which case $\int_V s^*(\alpha)(I) = \int_V \alpha(I) = \int_V s^*(\alpha)(I)$.

We observe that $\int_V \alpha(I)$ exists iff for each $V$ in $F$, $\alpha(I)$ exists. We take for granted the linearity and field-wise-additive properties of our integrals.

We state without proof a theorem of Kolmogoroff [4]:

**Theorem 2.K.1.** If $\int_V \alpha(I)$ exists, then $\int_V | \alpha(I) - \alpha(J) | = 0$.

Suppose each of \{ $a_i$ \}$_{i=1}^n$ and \{ $b_i$ \}$_{i=1}^n$ is a number sequence. We have the following two inequalities:

$$\min\{ a_1, \ldots, a_n \} + \min\{ b_1, \ldots, b_n \} \leq \min\{ a_1 + b_1, \ldots, a_n + b_n \},$$
The first of the above inequalities implies that if \( \{\beta_i\}_{i=1}^n \) is a sequence of elements of \( P_A^+ \) and \( E \) is a refinement of a subdivision \( D \) of a set \( V \) of \( F \), then

\[
0 \leq \sum_{i=1}^n \min \{\beta_1(I), \ldots, \beta_n(I)\} \leq \sum_{Q} \min \{\beta_1(I), \ldots, \beta_n(I)\},
\]

so that

\[
\int_V \min \{\beta_1(I), \ldots, \beta_n(I)\}
\]

exists.

An immediate consequence of Theorem 2.K.1 and the second of the above inequalities is the following corollary which we state without proof:

**Corollary 2.K.1.** If \( \{\beta_i\}_{i=1}^n \) is a sequence of elements of \( P_A^+ \) such that \( \int_U \beta_i(I) \) exists for \( i = 1, \ldots, n \), then

\[
\int_U \left| \min \{\beta_1(I), \ldots, \beta_n(I)\} - \min \left\{ \int_I \beta_1(J), \ldots, \int_I \beta_n(J) \right\} \right| = 0,
\]

so that if \( V \) is in \( F \), then

\[
\int_V \min \{\beta_1(I), \ldots, \beta_n(I)\}
\]

exists and is

\[
\int_V \min \left\{ \int_I \beta_1(I), \ldots, \int_I \beta_n(I) \right\}.
\]

If in subsequent statements, the existence of a given integral or its equivalence to a given integral is an immediate consequence of the statements of this section, the integral need only be written and the proof of existence or equivalence left to the reader.

3. **The representation theorem.** In this section we prove Theorem 3.1, as stated in the introduction.

**Proof of Theorem 3.1.** We see that there is a function \( \beta \) from \( F \) into \( \mathcal{R}^* \) such that if \( V \) is in \( F \), then

\[
| \xi(V) - \beta(V)\mu(V) | = \inf \{z \mid z = | \xi(V) - x\mu(V) |, x \in \mathcal{M} \}.
\]
Suppose $0 < c$. There is a finite subset $\{a_1, \ldots, a_n\}$ of $M$ such that if $x$ is in $M$, then $\min \{ |x - a_1|, \ldots, |x - a_n| \} < c/\left(4(\mu(U) + 1)\right)$.

We see that $\int_U \min \{ |\xi(I) - a_1\mu(I)|, \ldots, |\xi(I) - a_n\mu(I)| \} \, d\mu(I)$ exists, since $\int_U |\xi(I) - a_i\mu(I)| \, d\mu(I)$ exists for $i = 1, \ldots, n$.

There is a function $\gamma$ from $F$ into $\{a_1, \ldots, a_n\}$ such that if $I$ is in $F$, then

$$|\xi(I) - \gamma(I)\mu(I)| = \min \{ |\xi(I) - a_1\mu(I)|, \ldots, |\xi(I) - a_n\mu(I)| \}.$$

There is a subdivision $\mathcal{D}$ of $U$ such that if $E$ is a refinement of $\mathcal{D}$, then

$$\left| \int_U |\xi(I) - \gamma(I)\mu(I)| \, d\mu(I) \right| < c/4.$$

For each $I$ in $\mathcal{D}$ there is a subdivision $\mathcal{E}_I$ of $I$ and a function $n_I$ from $\mathcal{E}_I$ into $M$ such that $\sum_{\mathcal{E}_I} |\xi(J) - n_I(J)\mu(J)| < c/4N$, where $N$ is the number of elements of $\mathcal{D}$, and there is a function $\lambda_I$ from $\mathcal{E}_I$ into $\{a_1, \ldots, a_n\}$ such that for each $J$ in $\mathcal{E}_I$,

$$|\lambda_I(J) - n_I(J)| < c/\left(4(\mu(U) + 1)\right).$$

Therefore

$$\int_U |\xi(J) - \lambda_I(J)\mu(J)| \, d\mu(J) < c/4 + \sum_{\mathcal{D}} \sum_{\mathcal{E}_I} |\xi(J) - \lambda_I(J)\mu(J)|,$$

$$\leq c/4 + \sum_{\mathcal{D}} \sum_{\mathcal{E}_I} |\xi(J) - \lambda_I(J)\mu(J)|,$$

$$\leq c/4 + \sum_{\mathcal{D}} \sum_{\mathcal{E}_I} |\xi(J) - n_I(J)\mu(J)|,$$

$$+ \sum_{\mathcal{D}} \sum_{\mathcal{E}_I} |n_I(J) - \lambda_I(J)| \, d\mu(J)$$

$$< c/4 + N(c/4N) + \{c/\left(4(\mu(U) + 1)\right)\}\mu(U) < 3c/4.$$

For each $I$ in $\mathcal{D}$, there is a subdivision $\mathcal{E}_I'$ of $I$ such that $0 \leq s^*(|\xi - \beta\mu|)(I) - \sum_{\mathcal{E}_I'} |\xi(J) - \beta(J)\mu(J)| < c/16N$.

Now

$$\int_U s^*(|\xi - \beta\mu|)(I) \leq \sum_{\mathcal{D}} s^*(|\xi - \beta\mu|)(I),$$

$$\leq \sum_{\mathcal{D}} \left\{ c/16N + \sum_{\mathcal{E}_I} |\xi(J) - \beta(J)\mu(J)| \right\},$$

$$\leq c/16 + \sum_{\mathcal{D}} \sum_{\mathcal{E}_I} |\xi(J) - \gamma(J)\mu(J)|,$$

$$< c/16 + \int_U |\xi(I) - \gamma(J)\mu(J)| \, d\mu(J) + c/4,$$

$$< c/16 + 3c/4 + c/4 = 17c/16.$$
Therefore \( 0 \leq \int u s^* |x - \beta u| (I) \leq \int u s^* |x - \beta u| (I) = 0 \), so that 
\( \int u |\xi(I) - \beta(I)\mu(I)| \) exists and is 0, which we easily see implies that 
if \( V \) is in \( F \), then \( \int v \beta(I)\mu(I) \) exists and is \( \xi(V) \).

### 4. The characterization theorem.

**Definition** [1]: If \( \omega \) is in \( p^+ \), then the statement that \( \omega \) is \( \mu \)-refinement-unbounded means that if \( k \) is a positive number, then there is a subdivision \( \mathcal{D} \) of \( U \) such that if \( I \) is in a refinement of \( \mathcal{D} \) and \( \mu(I) \neq 0 \), then \( \omega(I) > k \).

We state a previous theorem of the author [2].

**Theorem 4.A.1.** Suppose \( \delta \) is in \( p^+ \) and that if each of \( c \) and \( k \) is a positive number, then there is a subdivision \( \mathcal{D} \) of \( U \) such that if \( \mathcal{C} \) is a refinement of \( \mathcal{D} \), then \( \sum \mathcal{C} \mu(I) < c \), where \( \mathcal{C}^* = \{ I \mid I \text{ in } \mathcal{C}, \delta(I) \leq k \} \). Then there is a \( \mu \)-refinement-unbounded element of \( p^+ \).

We now prove Theorem 4.1, as stated in the introduction.

**Proof of Theorem 4.1.** We first show that (1) implies (2).

Suppose (1) is true and \( \mathcal{M} \) is a bounded number set and \( \theta \) is a function from \( F \) into \( \mathcal{M} \) and \( \int u \theta(I)\mu(I) \) exists. Let \( \xi \) be the element of \( \mathcal{P} \) defined by \( \xi(V) = \int u \theta(I)\mu(I) \). Obviously \( \xi \) is in \( p_A \).

We now show that \( \xi \) is \( \mu \)-dense in \( \mathcal{M} \).

Suppose \( 0 < c \) and \( V \) is in \( F \). There is a subdivision \( \mathcal{D} \) of \( V \) such that if \( \mathcal{C} \) is a refinement of \( \mathcal{D} \), then \( \sum \mathcal{C} |\xi(I) - \theta(I)\mu(I)| < c/2 \). For each \( I \) in \( \mathcal{D} \), there is a number \( \lambda(I) \) in \( \mathcal{M} \) such that \( |\lambda(I) - \theta(I)| < c/[2(\mu(U) + 1)] \). This implies that

\[
\sum_{\mathcal{D}} |\xi(I) - \lambda(I)\mu(I)| \leq \sum_{\mathcal{D}} |\xi(I) - \theta(I)\mu(I)| + \sum_{\mathcal{D}} |\theta(I) - \lambda(I)| \mu(I)
\]

\[
< c/2 + \left\{ c/[2(\mu(U) + 1)] \right\} \mu(U) \leq c.
\]

Therefore \( \xi \) is \( \mu \)-dense in \( \mathcal{M} \) and therefore there is a function \( \phi \) from \( F \) into \( \mathcal{M} \) such that if \( V \) is in \( F \), then \( \int v \phi(I)\mu(I) \) exists and is \( \int v \phi(I)\mu(I) \).

Therefore (1) implies (2).

It is an immediate consequence of Theorem 3.1 that (2) implies (1).

We now show that (2) implies (3).

Suppose (2) is true. Let \( \mathcal{M} = \{ z | z = 1/q, q \text{ a positive integer} \} \). For each \( V \) in \( F \), let \( \theta(V) = 0 \). Obviously \( \int v \theta(I)\mu(I) = 0 \) for all \( V \) in \( F \). Since \( 0 \) is in \( \mathcal{M} \), it follows that there is a function \( \phi \) from \( F \) into \( \mathcal{M} \) such that if \( V \) is in \( F \), then \( \int v \phi(I)\mu(I) \) exists and is \( \int v \theta(I)\mu(I) \).

Now suppose that each of \( c \) and \( k \) is a positive number. There is a subdivision \( \mathcal{D} \) of \( U \) such that if \( \mathcal{C} \) is a refinement of \( \mathcal{D} \), then \( \sum \mathcal{C} \phi(I)\mu(I) < c/k \), so that \( \sum \mathcal{C} \mu(I) < c \), where \( \mathcal{C}^* = \{ I \mid I \text{ in } \mathcal{C}, \phi(I) \geq 1/k \} = \{ I \mid I \text{ in } \mathcal{C}, 1/\phi(I) \leq k \} \).
It follows that the function $1/\phi$ satisfies the hypothesis of Theorem 4.A.1, so that there is a $\mu$-refinement-unbounded element of $p^+$. Therefore (2) implies (3).

We now show that (3) implies (2). Suppose (3) is true, i.e. that there is a $\mu$-refinement-unbounded element $\omega$ of $p^+$, and $\mathcal{M}$ is a bounded number set and $\theta$ is a function from $F$ into $\mathcal{M}$ such that $\int_F \theta(I)\mu(I)$ exists.

There is a subdivision $\Xi^*$ of $U$ such that if $I$ is in a refinement of $\Xi^*$ and $\mu(I)\neq 0$, then $\omega(I) > 0$.

There is a function $\phi$ from $F$ into $\mathcal{M}$ such that if $I$ is in a refinement of $\Xi^*$ and $\mu(I)\neq 0$, then $|\theta(I) - \phi(I)| < 1/\omega(I)$.

Suppose $0 < c$ and $V$ is in $F$. There is a subdivision $\Xi$ of $V$ such that if $E$ is a refinement of $\Xi$, then $\left|\int_V \theta(I)\mu(I) - \sum_{E} \theta(I)\mu(I)\right| < c/2$. There is a refinement $\Xi'$ of $\Xi^*$ such that if $I$ is in a refinement of $\Xi'$ and $\mu(I)\neq 0$, then $\omega(I) > 2(\mu(U) + 1)/c$. There is a subdivision $\Xi''$ of $V$ which is a refinement of $\Xi$ and a subset of a refinement of $\Xi'$.

If $E$ is a refinement of $\Xi''$, then

$$\left|\int_V \theta(I)\mu(I) - \sum_{E} \phi(I)\mu(I)\right| \leq \left|\int_V \theta(I)\mu(I) - \sum_{E} \theta(I)\mu(I)\right|$$

$$+ \sum_{E} \left|\theta(I) - \phi(I)\right| \mu(I) < c/2 + \sum_{E^*} [\mu(I)/\omega(I)] \leq c/2$$

$$+ \{c/[2(\mu(U) + 1)]\} \mu(U) \leq c,$$

where $E^* = \{I | I$ in $E$, $\mu(I)\neq 0\}$.

Therefore $\int_V \phi(I)\mu(I)$ exists and is $\int_V \theta(I)\mu(I)$. Therefore (3) implies (2). Therefore (1), (2) and (3) are equivalent.

References


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